

ON THE WAVE-BREAKING PHENOMENA AND GLOBAL EXISTENCE FOR THE GENERALIZED PERIODIC CAMASSA-HOLM EQUATION

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ABSTRACT. Considered herein is the initial-value problem for the generalized periodic Camassa-Holm equation which is related to the Camassa-Holm equation and the Hunter-Saxton equation. Sufficient conditions guaranteeing the development of breaking waves in finite time are demonstrated. On the other hand, the existence of strong permanent waves is established with certain initial profiles depending on the linear dispersive parameter in a range of the Sobolev spaces. Moreover, the admissible global weak solution in the energy space is obtained.

Keywords: Generalized periodic Camassa-Holm equation, Wave-breaking, Global existence

AMS Subject Classification (2000): 35B30, 35G25

1. INTRODUCTION

We study here the initial-value problem associated with the generalized periodic Camassa-Holm (μ -CH) equation [30], namely,

$$\begin{cases} \mu(u_t) - u_{xxt} + 2\mu(u)u_x + 2\kappa u_x = 2u_x u_{xx} + uu_{xxx}, & t > 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t, x), & t \geq 0, \quad x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where $u(t, x)$ is a time-dependent function on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{\mathbb{S}} u(t, x) dx$ denotes its mean, the parameter $\kappa \in \mathbb{R}$. Obviously, if $\mu(u) = 0$, which implies that $\mu(u_t) = 0$, then this equation reduces to the Hunter-Saxton (HS) equation [25], which is also a short wave limit of the Camassa-Holm (CH) equation [1, 5, 13, 23]. Equivalently, the initial value problem (1.1) can be rewritten as the following mixed hyperbolic-elliptic type system.

$$\begin{cases} u_t + uu_x + \partial_x P = 0, & t > 0, \quad x \in \mathbb{R}, \\ (\mu - \partial_x^2)P = 2\mu(u)u + \frac{1}{2}u_x^2 + 2\kappa u, & t > 0, \quad x \in \mathbb{R} \\ u(t, x+1) = u(t, x), & t \geq 0, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}. \end{cases} \quad (1.2)$$

With $m = (\mu - \partial_x^2)u$, the first equation in (1.1) may be read as

$$m_t + um_x + 2mu_x + 2\kappa u_x = 0. \quad (1.3)$$

It is known that the Camassa-Holm equation is one of the following family of equations with the parameter $\lambda = 2$

$$m_t + um_x + \lambda u_x m + 2\kappa u_x = 0, \quad (1.4)$$

with $m = Au$ and $A = 1 - \partial_x^2$, the parameter $\kappa \in \mathbb{R}$. The family of equations are believed to be integrable [5, 17] only for $\lambda = 2$ and $\lambda = 3$.

It is observed that the μ -CH equation is the corresponding μ -version of the family given by (1.4) with $m = Au$, $A = \mu - \partial_x^2$, and the parameter $\lambda = 2$.

It is clear that the closest relatives of the μ -CH equation are the Camassa-Holm equation with $A = 1 - \partial_x^2$

$$u_t - u_{txx} + 3uu_x + 2\kappa u_x = 2u_x u_{xx} + uu_{xxx},$$

and the equation with $A = -\partial_x^2$

$$-u_{txx} + 2\kappa u_x = 2u_x u_{xx} + uu_{xxx}. \quad (1.5)$$

It is noted that when $\kappa = 0$, (1.5) becomes the Hunter-Saxton equation, while if $\kappa \neq 0$, (1.5) is a short wave limit of the Camassa-Holm (CH) equation, which is an equation in the Dym hierarchy and admits a new class of soliton solutions (called umbilic solitons) [1].

Both of the CH equation and the HS equation have attracted a lot of attention among the integrable systems and the PDE communities. The Camassa-Holm equation was introduced in [5] as a shallow water approximation and has a bi-Hamiltonian structure [23], whose relevance for water waves was established in [13]. The Hunter-Saxton equation firstly appeared in [25] as an asymptotic equation for rotators in liquid crystals. Recently, it was claimed in [18] that the equation might be relevant to the modeling of tsunamis, also see the discussion in [12].

The Camassa-Holm equation is a completely integrable system with a bi-Hamiltonian structure and hence it possesses an infinite sequence of conservation laws [5, 23], see [14] for the periodic case. When $\kappa = 0$, it admits soliton-like solutions (called peakons) in both periodic and non-periodic setting [5] and the multi-soliton or infinite-soliton solutions consisting of a train of peaked solitary waves or ‘peakons’ [5, 6]. These peakons are weak solutions in the distributional sense and shown to be stable [6, 15, 16, 19, 20]. The Camassa-Holm equation describes geodesic flows on the infinite dimensional group $\mathcal{D}^s(\mathbb{S})$ of orientation-preserving diffeomorphisms of the unit circle \mathbb{S} of Sobolev class H^s and endowed with a right-invariant metric by the H^1 inner product [31, 35]. The Hunter-Saxton equation also describes the geodesic flow on the homogeneous space of the group $\mathcal{D}^s(\mathbb{S})$ modulo the subgroup of rigid rotations $Rot(\mathbb{S}) \simeq \mathbb{S}$ equipped with the \dot{H}^1 right-invariant metric [32] at the identity

$$\langle u, v \rangle_{\dot{H}^1} = \int_{\mathbb{S}} u_x v_x dx.$$

The Hunter-Saxton equation possesses a bi-Hamiltonian structure and is formally integrable [26].

Another remarkable property of the Camassa-Holm equation is the presence of breaking waves (i.e. the solution remains bounded while its slope becomes unbounded in finite time [39]) [5, 8, 9, 10, 14, 34]. Wave breaking is one of the most intriguing long-standing problems of water wave theory [39]. It is worth pointing out that Bressan and Constantin proved that the solutions to the Camassa-Holm equation can be uniquely continued after wave-breaking as either global conservative or global dissipative weak solution in [2] and [3], respectively. It is noted that Xin and Zhang obtained the existence of a global-in-time weak solution to the Camassa-Holm equation in the energy space [41], where authors basically follow the approach in [45] to study the viscous approximate solutions to the Camassa-Holm equation.

The μ -CH was introduced by Khesin, Lenells and Misiolek [30] (also called μ -HS equation). Similar to the HS equation [25], the μ -CH equation describes the propagation of

weakly nonlinear orientation waves in a massive nematic liquid crystal with external magnetic field and self-interaction. Here, the solution $u(t, x)$ of the μ -CH equation represents the director field of a nematic liquid crystal, x is a space variable in a reference frame moving with the linearized wave velocity, and t is a slow time variable. Nematic liquid crystals are fields consisting of long rigid molecules. The μ -CH equation is also an Euler equation on $\mathcal{D}^s(\mathbb{S})$ (the set of circle diffeomorphism of the Sobolev class H^s) and it describes the geodesic flow on $\mathcal{D}^s(\mathbb{S})$ with the right-invariant metric given at the identity by the inner product [30]

$$\langle u, v \rangle = \mu(u)\mu(v) + \int_{\mathbb{S}} u_x v_x dx.$$

It was shown in [30] that the μ -CH equation is formally integrable and can be viewed as the compatibility condition between

$$\psi_{xx} = \xi(m + \kappa)\psi \quad \text{and} \quad \psi_t = \left(\frac{1}{2\xi} - u\right)\psi_x + \frac{1}{2}u_x\psi,$$

where $\xi \in \mathbb{C}$ is a spectral parameter and $m = \mu(u) - u_{xx}$.

On the other hand, the μ -CH equation admits bi-Hamiltonian structure and infinite hierarchy of conservation laws. The first few conservation laws in the hierarchy are

$$H_0 = \int_{\mathbb{S}} m dx, \quad H_1 = \frac{1}{2} \int_{\mathbb{S}} mu dx, \quad H_2 = \int_{\mathbb{S}} \left(\mu(u)u^2 + \kappa u^2 + \frac{1}{2}u u_x^2 \right) dx.$$

It is noted that the Hunter-Saxton equation does not have any bounded traveling-wave solutions at all, while the μ -CH equation admits traveling waves that can be regarded as the appropriate candidates for solitons. It is shown in [30, 33] that when $\kappa = 0$, the μ -CH equation admits not only periodic one-peakon solution $u(t, x) = \varphi(x - ct)$ where

$$\varphi(x) = \frac{c}{26}(12x^2 + 23)$$

for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and φ is extended periodically to the real line, but also the multi-peakons of the form

$$u = \sum_{i=1}^N p_i(t)g(x - q_i(t)),$$

where $g(x) = \frac{1}{2}x(x - 1) + \frac{13}{12}$ is the Green function of the operator $(\mu - \partial_x^2)^{-1}$.

Remark 1.1. The operator $A = \mu - \partial_x^2$ is elliptic and an isomorphism between $H^s(\mathbb{S})$ and $H^{s-2}(\mathbb{S})$ since

$$(\widehat{Au})(k) = \begin{cases} (1 + k^2)\widehat{u}(k), & \text{for } k = 0, \\ k^2\widehat{u}(k), & \text{for } k \in \mathbb{Z} \setminus \{0\}, \end{cases}$$

where we denote the Fourier transform of a function f in the torus \mathbb{S} by $\widehat{f}(k)$ with the frequency $k \in \mathbb{Z}$. In particular, if u is constant, then $Au = u = A^{-1}u$.

According to the Green function of the operator $A^{-1} = (\mu - \partial_x^2)^{-1}$ (that is, $g(x) = \frac{1}{2}x(x-1) + \frac{13}{12}$), the inverse $v = A^{-1}w$ can be given explicitly by

$$\begin{aligned} v(x) &= \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12} \right) \mu(w) + \left(x - \frac{1}{2} \right) \int_0^1 \int_0^y w(s) ds dy \\ &\quad - \int_0^x \int_0^s w(r) dr ds + \int_0^1 \int_0^y \int_0^s w(r) dr ds dy \\ &= \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12} \right) \mu(w) + \left(x - \frac{1}{2} \right) \int_0^1 \int_0^y w(s) ds dy + \int_0^1 \int_x^y \int_0^s w(r) dr ds dy. \end{aligned} \quad (1.6)$$

Since A^{-1} and ∂_x commute, the following identities hold

$$A^{-1} \partial_x w(x) = \left(x - \frac{1}{2} \right) \int_0^1 w(x) dx - \int_0^x w(y) dy + \int_0^1 \int_0^x w(y) dy dx,$$

and

$$A^{-1} \partial_x^2 w(x) = -w(x) + \int_0^1 w(x) dx. \quad (1.7)$$

Thanks to (1.6), we can read explicitly the formulation of P in (1.2) as

$$\begin{aligned} P &= \left(x^2 - x + \frac{13}{6} \right) (\mu(u) + \kappa) \mu(u) + (2x - 1) (\mu(u) + \kappa) \int_0^1 \int_0^y u(s) ds dy \\ &\quad + 2(\mu(u) + \kappa) \int_0^1 \int_x^y \int_0^s u(r) dr ds dy + \frac{1}{2} \int_0^1 \int_x^y \int_0^s (\partial_x u)^2(r) dr ds dy \\ &\quad + \frac{1}{2} \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12} \right) \|\partial_x u\|_{L^2}^2 + \frac{1}{2} \left(x - \frac{1}{2} \right) \int_0^1 \int_0^y (\partial_x u)^2(s) ds dy, \end{aligned}$$

which leads to

$$\begin{aligned} \partial_x P &= \left(\frac{1}{2}x - \frac{1}{4} \right) (2\mu(u)(\mu(u) + \kappa) + \|\partial_x u\|_{L^2}^2) + \frac{1}{2} \int_0^1 \int_0^y (\partial_x u)^2(s) ds dy \\ &\quad + 2(\mu(u) + \kappa) \left(\int_0^1 \int_0^x u(y) dy dx - \int_0^x u(y) dy \right) - \frac{1}{2} \int_0^x (\partial_x u)^2(y) dy. \end{aligned} \quad (1.8)$$

Note that $H^s \hookrightarrow Lip$ for $s > \frac{3}{2}$. From the theory of the transport equation point of view, one may define a strong solution to (1.2) as follows.

Definition 1.1. If $u \in C([0, T], H^s(\mathbb{S})) \cap C^1([0, T], H^{s-1}(\mathbb{S}))$ with $s > \frac{3}{2}$ satisfies (1.2), then u is called a strong solution to (1.2). If u is a strong solution on $[0, T)$ for every $T > 0$, then it is called global strong solution to (1.2).

One of our goals in this paper is concerned with the existence of a global weak solution in the energy space H^1 , which is motivated by the work in [41].

Definition 1.2. A continuous function $u = u(t, x)$ is said to be an admissible global weak solution to the initial-value problem (1.2) if

(i) $u(t, x) \in C(\mathbb{R}^+ \times \mathbb{S}) \cap L^\infty(\mathbb{R}^+, H^1(\mathbb{S}))$ and

$$\mu(u) = \mu(u_0) \quad \text{and} \quad \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{S})} \leq \|\partial_x u_0(\cdot)\|_{L^2(\mathbb{S})} \quad \forall \quad t > 0; \quad (1.9)$$

(ii) $u(t, x)$ satisfies the equations in (1.2) in the sense of distributions and takes on the initial data pointwise.

Our main results of the present paper are Theorems 3.1- 3.4 (wave-breaking), Theorems 4.1-4.2(Global strong solution), and Theorem 5.1(Global weak solution).

The remainder of the paper is organized as follows. In Section 2, some *a priori* estimates and basic properties on its strong solutions to the μ -CH equation are recalled and derived, which are constantly used in the whole paper. In Section 3, the results of blow-up to strong solutions are established in details. It is shown that the solutions of the μ -CH equation can only have singularities which correspond to wave breaking (Theorems 3.1-3.4). Two sufficient conditions for the existence of global strong solutions (Theorems 4.1-4.2) are specified in Section 4. The existence of an admissible global weak solution in the energy space H^1 (Theorem 5.1) is demonstrated in the last section, Section 5.

Notations. Throughout this paper, we identify all spaces of periodic functions with function spaces over the unit circle \mathbb{S} in \mathbb{R}^2 , i. e. $\mathbb{S} = \mathbb{R}/\mathbb{Z}$. Since all space of functions are over \mathbb{S} , for simplicity, we drop \mathbb{S} in our notations of function spaces if there is no ambiguity. For a given Banach space Z , we denote its norm by $\|\cdot\|_Z$.

2. PRELIMINARIES

In the following, we establish some *a priori* estimates for the μ -CH equation. Recall that the first two conserved quantities of the μ -CH equation are

$$H_0 = \int_{\mathbb{S}} m \, dx = \int_{\mathbb{S}} (\mu(u) - u_{xx}) \, dx = \mu(u(t)),$$

and

$$H_1 = \frac{1}{2} \int_{\mathbb{S}} mu \, dx = \frac{1}{2} \mu^2(u(t)) + \frac{1}{2} \int_{\mathbb{S}} u_x^2(t, x) dx.$$

It is easy to see that $\mu(u(t))$ and $\int_{\mathbb{S}} u_x^2(t, x) dx$ are conserved in time [30]. Thus

$$\mu(u_t) = 0.$$

For the sake of convenience, let

$$\mu_0 = \mu(u_0) = \mu(u(t)) = \int_{\mathbb{S}} u(t, x) dx \quad (2.1)$$

and

$$\mu_1 = \left(\int_{\mathbb{S}} u_x^2(0, x) dx \right)^{\frac{1}{2}} = \left(\int_{\mathbb{S}} u_x^2(t, x) dx \right)^{\frac{1}{2}}. \quad (2.2)$$

Then μ_0 and μ_1 are constants and independent of time t .

Lemma 2.1. [9] *If $f \in H^3(\mathbb{S})$ is such that $\int_{\mathbb{S}} f(x) \, dx = a_0/2$, then for every $\varepsilon > 0$, we have*

$$\max_{x \in \mathbb{S}} f^2(x) \leq \frac{\varepsilon + 2}{24} \int_{\mathbb{S}} f_x^2(x) dx + \frac{\varepsilon + 2}{4\varepsilon} a_0^2.$$

Remark 2.1. *Since H^3 is dense in H^1 , Lemma 2.1 also holds for every $f \in H^1(\mathbb{S})$. Moreover, if $\int_{\mathbb{S}} f(x) \, dx = 0$, from the deduction of this lemma we arrive at the following inequality*

$$\max_{x \in \mathbb{S}} f^2(x) \leq \frac{1}{12} \int_{\mathbb{S}} f_x^2(x) dx, \quad x \in \mathbb{S}, \quad f \in H^1(\mathbb{S}). \quad (2.3)$$

Lemma 2.2. [4] *For every $f(x) \in H^1(a, b)$ periodic and with zero average, i.e. such that $\int_a^b f(x) dx = 0$, we have*

$$\int_a^b f^2(x) dx \leq \left(\frac{b-a}{2\pi} \right)^2 \int_a^b |f'(x)|^2 dx,$$

and equality holds if and only if

$$f(x) = A \cos \left(\frac{2\pi x}{b-a} \right) + B \sin \left(\frac{2\pi x}{b-a} \right).$$

Note that

$$\int_{\mathbb{S}} (u(t, x) - \mu_0) dx = \mu_0 - \mu_0 = 0.$$

By Lemma 2.1, we find that

$$\max_{x \in \mathbb{S}} [u(t, x) - \mu_0]^2 \leq \frac{1}{12} \int_{\mathbb{S}} u_x^2(t, x) dx = \frac{1}{12} \int_{\mathbb{S}} u_x^2(0, x) dx = \frac{1}{12} \mu_1^2. \quad (2.4)$$

From the above estimate, we find that the amplitude of the wave remains bounded in any time, that is,

$$\|u(t, \cdot)\|_{L^\infty} - |\mu_0| \leq \|u(t, \cdot) - \mu_0\|_{L^\infty} \leq \frac{\sqrt{3}}{6} \mu_1,$$

and so

$$\|u(t, \cdot)\|_{L^\infty} \leq |\mu_0| + \frac{\sqrt{3}}{6} \mu_1. \quad (2.5)$$

While thanks to Lemma 2.2, we have

$$\int_{\mathbb{S}} [u(t, x) - \mu_0]^2 dx \leq \frac{1}{4\pi^2} \int_{\mathbb{S}} u_x^2(t, x) dx = \frac{1}{4\pi^2} \int_{\mathbb{S}} u_x^2(0, x) dx = \frac{1}{4\pi^2} \mu_1^2. \quad (2.6)$$

Therefore, one gets from (2.6) that

$$\begin{aligned} \|u(t, x)\|_{L^2}^2 &= \int_{\mathbb{S}} u^2(t, x) dx = \int_{\mathbb{S}} [(u - \mu_0)^2 + 2\mu_0 u - \mu_0^2](t, x) dx \\ &\leq \frac{1}{4\pi^2} \mu_1^2 + \mu_0^2. \end{aligned} \quad (2.7)$$

It then follows that

$$\|u(t, \cdot)\|_{H^1}^2 = \int_{\mathbb{S}} u^2(t, x) dx + \int_{\mathbb{S}} u_x^2(t, x) dx \leq \frac{1 + 4\pi^2}{4\pi^2} \mu_1^2 + \mu_0^2.$$

Let us first state the following local well-posedness result of (1.2), which was obtained in [30] and [33] (up to a slight modification, the proof is omitted).

Proposition 2.1. *Let $u_0 \in H^s(\mathbb{S})$, $s > 3/2$. Then there exist a maximal $T = T(u_0) > 0$ and a unique strong solution u to (1.2) such that*

$$u = u(\cdot, u_0) \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $u_0 \mapsto u(\cdot, u_0) : H^s(\mathbb{S}) \rightarrow C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}))$ is continuous.

Remark 2.2. *The maximal T in Proposition 2.1 can be chosen independent of s in the following sense. If $u = u(\cdot, u_0) \in C([0, T), H^s) \cap C^1([0, T), H^{s-1})$ to (1.2) and $u_0 \in H^{s'}$ for some $s' \neq s$, $s' > \frac{3}{2}$, then $u \in C([0, T), H^{s'}) \cap C^1([0, T), H^{s'-1})$ and with the same T . In particular, if $u_0 \in H^\infty = \bigcap_{s \geq 0} H^s$, then $u \in C([0, T), H^\infty)$ (see [22])*

for the details, or [24, 37] for an adaptation of the Kato method [28] to the proof of this statement for the (generalized) Camassa-Holm equation).

Let us now consider the following differential equation

$$\begin{cases} q_t = u(t, q), & t \in [0, T), \\ q(0, x) = x, & x \in \mathbb{R}. \end{cases} \quad (2.8)$$

Applying classical results in the theory of ordinary differential equations, we have the following properties of q which are crucial in the proof of global existence.

Lemma 2.3. *Let $u_0 \in H^s(\mathbb{S})$, $s > \frac{3}{2}$, and let $T > 0$ be the maximal existence time of the corresponding strong solution u to (1.2). Then Eq.(2.8) has a unique solution $q \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$ such that the map $q(t, \cdot)$ is an increasing diffeomorphism of \mathbb{R} with*

$$q_x(t, x) = \exp \left(\int_0^t u_x(s, q(s, x)) ds \right) > 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Furthermore, setting $m = \mu(u) - u_{xx}$, we have

$$(m(t, q(t, x)) + \kappa) q_x^2(t, x) = m_0(x) + \kappa, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

Proof. Since $u \in C^1([0, T), H^{s-1}(\mathbb{S}))$ and $H^s(\mathbb{S}) \hookrightarrow C^1(\mathbb{S})$, we see that both functions $u(t, x)$ and $u_x(t, x)$ are bounded, Lipschitz in the space variable x , and of class C^1 in time. Therefore, for fixed $x \in \mathbb{R}$, (2.8) is an ordinary differential equation. Then well-known classical results in the theory of ordinary differential equation yield that (2.8) has a unique solution $q(t, x) \in C^1([0, T) \times \mathbb{R}, \mathbb{R})$.

Differentiation of (2.8) with respect to x yields

$$\begin{cases} \frac{d}{dt} q_x = u_x(t, q) q_x, & t \in [0, T), \\ q_x(0, x) = 1, & x \in \mathbb{R}. \end{cases} \quad (2.9)$$

The solution to (2.9) is given by

$$q_x(t, x) = \exp \left(\int_0^t u_x(s, q(s, x)) ds \right), \quad (t, x) \in [0, T) \times \mathbb{R}. \quad (2.10)$$

For every $T' < T$, it follows from the Sobolev imbedding theorem that

$$\sup_{(s, x) \in [0, T') \times \mathbb{R}} |u_x(s, x)| < \infty.$$

We infer from (2.10) that there exists a constant $K > 0$ such that $q_x(t, x) \geq e^{-Kt}$, $(t, x) \in [0, T) \times \mathbb{R}$, which implies that the map $q(t, \cdot)$ is an increasing diffeomorphism of \mathbb{R} with

$$q_x(t, x) = \exp \left(\int_0^t u_x(s, q(s, x)) ds \right) > 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

On the other hand, combining (2.9) with (1.3), we have

$$\begin{aligned} \frac{d}{dt} ((m(t, q(t, x)) + \kappa) q_x^2(t, x)) &= (m_t + m_x q_x) q_x^2(t, x) + 2(m + \kappa) q_x q_{xt} \\ &= q_x^2(m_t + m_x u + 2u_x m + 2\kappa u_x) = 0. \end{aligned}$$

So,

$$(m(t, q(t, x)) + \kappa) q_x^2(t, x) = m_0(x) + \kappa, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.$$

This completes the proof of Lemma 2.3. \square

Remark 2.3. Lemma 2.3 shows that, if $m_0 + \kappa = \mu(u_0) - u_{0xx} + \kappa$ does not change sign, then $m(t) + \kappa$ ($\forall t$) will not change sign, as long as $m(t)$ exists.

Remark 2.4. Since $q(t, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is a diffeomorphism of the line for every $t \in [0, T)$, the L^∞ -norm of any function $v(t, \cdot) \in L^\infty$, $t \in [0, T)$ is preserved under the family of diffeomorphisms $q(t, \cdot)$ with $t \in [0, T)$, that is,

$$\|v(t, \cdot)\|_{L^\infty} = \|v(t, q(t, \cdot))\|_{L^\infty}, \quad t \in [0, T).$$

In [30] and [33], the authors also showed that the μ -CH equation admits global (in time) solutions and blow-up solutions. It is our purpose here to derive the precise wave-breaking scenarios and determine the initial conditions guaranteeing the blow-up of strong solutions to the initial-value problem (1.1), which will significantly improve the results in [30] and [33].

As long as the solution u to (1.2) is defined, we set

$$m_1(t) = \min_{x \in \mathbb{S}} [u_x(t, x)], \quad \text{and} \quad m_2(t) = \max_{x \in \mathbb{S}} [u_x(t, x)] \quad (2.11)$$

and further $x_1(t) \in \mathbb{S}$ and $x_2(t) \in \mathbb{S}$ are points where these extrema are attained, i.e., $m_i(t) = u_x(t, x_i(t))$, $i = 1, 2$. We will make use of the following lemma.

Lemma 2.4. [11] Let $[0, T)$ be the maximal interval of existence of the solution $u(t, x)$ of (1.2) with the initial data $u_0 \in H^s$, $s > \frac{3}{2}$, as given by Proposition 2.1. Then the functions $m_i(t)$, $i = 1, 2$, are absolutely continuous on $(0, T)$ with

$$\frac{dm_i}{dt} = u_{xt}(t, x_i(t)), \quad \text{a.e. on } (0, T).$$

3. WAVE-BREAKING MECHANISM

In this section, we derive some sufficient conditions for the breaking waves to the initial-value problem (1.2). We first state the precise wave-breaking scenario for the problem (1.2) in the following, which was obtained in [22] (up to a slight modification).

Proposition 3.1. Let $u_0 \in H^s(\mathbb{S})$, $s > 3/2$, and $u(t, x)$ be the solution of the initial-value problem (1.2) with life-span T . Then T is finite if and only if

$$\liminf_{t \uparrow T} \left(\inf_{x \in \mathbb{S}} u_x(t, x) \right) = -\infty.$$

In what follows, we establish some sufficient conditions guaranteeing the development of singularities by means of the wave-breaking scenario. We are now in a position to give the first wave-breaking result for the μ -CH equation.

Theorem 3.1. Let $u_0 \in H^s(\mathbb{S})$, $s > 3/2$ and $T > 0$ be the maximal time of existence of the corresponding solution $u(t, x)$ to (1.2) with the initial data u_0 . If $(\sqrt{3}/\pi)|\mu_0 + \kappa| < \mu_1$, where μ_0 and μ_1 are defined in (2.1) and (2.2), then the corresponding solution $u(t, x)$ to (1.2) associated with the μ -CH equation must blow up in finite time T with

$$0 < T \leq \inf_{\alpha \in I} \left(\frac{6}{1 - 6\alpha} + 4\pi^2 \alpha \frac{1 + |\int_{\mathbb{S}} u_{0x}^3(x) dx|}{6\pi^2 \alpha \mu_1^4 - 3(\mu_0 + \kappa)^2 \mu_1^2} \right)$$

where $I = \left(\frac{(\mu_0 + \kappa)^2}{2\pi^2 \mu_1^2}, \frac{1}{6} \right)$, such that

$$\liminf_{t \uparrow T} \left(\inf_{x \in \mathbb{S}} u_x(t, x) \right) = -\infty.$$

Proof. Thanks to Remark 2.2, it suffices to consider the case $s = 3$. Differentiating the first equation in (1.2) with respect to x yields

$$u_{tx} + u_x^2 + uu_{xx} + A^{-1}\partial_x^2 \left(2u\mu_0 + \frac{1}{2}u_x^2 + 2\kappa u \right) = 0. \quad (3.1)$$

In view of (2.1), (2.2) and (1.7), we have

$$u_{tx} = -\frac{1}{2}u_x^2 - uu_{xx} + 2u(\mu_0 + \kappa) - 2\mu_0^2 - \frac{1}{2}\mu_1^2 - 2\kappa\mu_0. \quad (3.2)$$

Multiplying (3.2) by $3u_x^2$ and integrating on \mathbb{S} with respect to x , we obtain for any $t \in [0, T)$ that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{S}} u_x^3 dx &= \int_{\mathbb{S}} 3u_x^2 u_{xt} dx = -\frac{3}{2} \int_{\mathbb{S}} u_x^4 dx - \int_{\mathbb{S}} 3uu_x^2 u_{xx} dx \\ &\quad + 6(\mu_0 + \kappa) \int_{\mathbb{S}} (u - \mu_0) u_x^2 dx - \frac{3}{2} \left(\int_{\mathbb{S}} u_x^2 dx \right)^2 \\ &= -\frac{1}{2} \int_{\mathbb{S}} u_x^4 dx - \frac{3}{2} \mu_1^4 + 6(\mu_0 + \kappa) \int_{\mathbb{S}} (u - \mu_0) u_x^2 dx. \end{aligned} \quad (3.3)$$

On the other hand, it follows from Lemma 2.2 for any $\alpha > 0$ that

$$\begin{aligned} (\mu_0 + \kappa) \int_{\mathbb{S}} (u - \mu_0) u_x^2 dx &\leq |\mu_0 + \kappa| \left(\int_{\mathbb{S}} (u - \mu_0)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{S}} u_x^4 dx \right)^{\frac{1}{2}} \\ &\leq \frac{\alpha}{2} \int_{\mathbb{S}} u_x^4 dx + \frac{(\mu_0 + \kappa)^2}{2\alpha} \int_{\mathbb{S}} (u - \mu_0)^2 dx \\ &\leq \frac{\alpha}{2} \int_{\mathbb{S}} u_x^4 dx + \frac{(\mu_0 + \kappa)^2}{8\pi^2\alpha} \int_{\mathbb{S}} u_x^2 dx. \end{aligned}$$

Therefore we deduce that

$$\frac{d}{dt} \int_{\mathbb{S}} u_x^3 dx \leq \left(3\alpha - \frac{1}{2} \right) \int_{\mathbb{S}} u_x^4 dx - \frac{3}{2} \mu_1^4 + \frac{3}{4\pi^2\alpha} (\mu_0 + \kappa)^2 \mu_1^2. \quad (3.4)$$

By the assumption of the theorem, we know that $(\mu_0 + \kappa)^2 / (2\pi^2 \mu_1^2) < 1/6$. Let $\alpha > 0$ satisfy

$$\frac{(\mu_0 + \kappa)^2}{2\pi^2 \mu_1^2} < \alpha < \frac{1}{6}.$$

This in turn implies that

$$c_1 := \frac{1}{2} - 3\alpha > 0 \quad \text{and} \quad c_2 := \frac{3}{2} \mu_1^4 - \frac{3}{4\pi^2\alpha} (\mu_0 + \kappa)^2 \mu_1^2 > 0.$$

Hence, applying Hölder's inequality to (3.4) yields

$$\frac{d}{dt} \int_{\mathbb{S}} u_x^3 dx \leq -c_1 \int_{\mathbb{S}} u_x^4 dx - c_2 \leq -c_1 \left(\int_{\mathbb{S}} u_x^3 dx \right)^{\frac{4}{3}} - c_2.$$

Let $V(t) = \int_{\mathbb{S}} u_x^3(t, x) dx$ with $t \in [0, T)$. Then the above inequality can be rewritten as

$$\frac{d}{dt} V(t) \leq -c_1 (V(t))^{\frac{4}{3}} - c_2 \leq -c_2 < 0, \quad t \in [0, T). \quad (3.5)$$

This implies that $V(t)$ decreases strictly in $[0, T)$. Let $t_1 = (1 + |V(0)|)/c_2$. One can assume $t_1 < T$. Otherwise, $T \leq t_1 < \infty$ and the theorem is proved. It then follows from

(3.5) that

$$V(t) \leq \left[\frac{3}{c_1(t - t_1) - 3} \right]^3 \rightarrow -\infty, \quad \text{as } t \rightarrow t_1 + \frac{3}{c_1}.$$

On the other hand, we have

$$V(t) = \int_{\mathbb{S}} u_x^3 dx \geq \inf_{x \in \mathbb{S}} u_x(t, x) \int_{\mathbb{S}} u_x^2 dx = \mu_1^2 \inf_{x \in \mathbb{S}} u_x(t, x).$$

This then implies that $0 < T \leq t_1 + 3/c_1$ such that

$$\liminf_{t \uparrow T} \left(\inf_{x \in \mathbb{S}} u_x(t, x) \right) = -\infty.$$

This completes the proof of Theorem 3.1. \square

In the case $(\sqrt{3}/\pi)|\mu_0 + \kappa| \geq \mu_1$, we have the following wave-breaking result.

Theorem 3.2. *Let $u_0 \in H^s(\mathbb{S})$, $s > 3/2$ and $T > 0$ be the maximal time of existence of the corresponding solution $u(t, x)$ to (1.2) with the initial data u_0 . If $(\sqrt{3}/\pi)|\mu_0 + \kappa| \geq \mu_1$ and*

$$\inf_{x \in \mathbb{S}} u'_0(x) < -\sqrt{2\mu_1 \left(\frac{\sqrt{3}}{3}|\mu_0 + \kappa| - \frac{1}{2}\mu_1 \right)} \equiv -K,$$

where $u'_0(x)$ is the derivative of $u_0(x)$ with respect to x , then the corresponding solution $u(t, x)$ to (1.2) blows up in finite time T with

$$0 < T \leq \frac{\inf_{x \in \mathbb{S}} u'_0(x)}{K^2 - (\inf_{x \in \mathbb{S}} u'_0(x))^2},$$

such that

$$\liminf_{t \uparrow T} \left(\inf_{x \in \mathbb{S}} u_x(t, x) \right) = -\infty.$$

Proof. As discussed above, it suffices to consider the case $s = 3$. Note that the assumption $(\sqrt{3}/\pi)|\mu_0 + \kappa| \geq \mu_1$ implies that $(2/\sqrt{3})|\mu_0 + \kappa| > \mu_1$. Therefore the non-negative constant K is well-defined.

By Lemma 2.4, there is $x_0 \in \mathbb{S}$ such that $u'_0(x_0) = \inf_{x \in \mathbb{S}} u'_0(x)$. Define $w(t) = u_x(t, q(t, x_0))$, where $q(t, x_0)$ is the flow of $u(t, q(t, x_0))$. Then

$$\frac{d}{dt}w(t) = (u_{tx} + u_{xx}q_t)(t, q(t, x_0)) = (u_{tx} + uu_{xx})(t, q(t, x_0)).$$

Substituting $(t, q(t, x_0))$ into (3.2) and using (2.3), we obtain

$$\begin{aligned} \frac{d}{dt}w(t) &= -\frac{1}{2}w^2(t) + 2(\mu_0 + \kappa)u(t, q(t, x_0)) - 2\mu_0(\mu_0 + \kappa) - \frac{1}{2}\mu_1^2 \\ &= -\frac{1}{2}w^2(t) + 2(\mu_0 + \kappa)[u(t, q(t, x_0)) - \mu_0] - \frac{1}{2}\mu_1^2, \end{aligned}$$

which together with (2.4) implies that

$$\frac{d}{dt}w(t) \leq -\frac{1}{2}w^2(t) + \mu_1 \left(\frac{\sqrt{3}}{3}|\mu_0 + \kappa| - \frac{1}{2}\mu_1 \right) = -\frac{1}{2}w^2(t) + \frac{1}{2}K^2. \quad (3.6)$$

By the assumption $w(0) = u'_0(x_0) < -K$, we have $w^2(0) > K^2$. We now claim that $w(t) < -K$ holds for any $t \in [0, T)$. In fact, assuming the contrary would, in view of

$w(t)$ being continuous, ensure the existence of $t_0 \in (0, T)$ such that $w^2(t) > K^2$ for $t \in [0, t_0]$ but $w^2(t_0) = K^2$. Combining this with (3.6) would give

$$\frac{d}{dt}w(t) < 0 \quad \text{a.e. on} \quad [0, t_0]. \quad (3.7)$$

Since $w(t)$ is absolutely continuous on $[0, t_0]$, an integration of this inequality would give the following inequality and we get the contradiction

$$w(t_0) < w(0) = u'_0(x_0) < -K.$$

This proves the previous claim. Therefore, we get $\frac{d}{dt}w(t) < 0$ on $[0, T)$, which implies that $w(t)$ is strictly decreasing on $[0, T)$. Set

$$\delta := 1 - \left(\frac{K}{u'_0(x_0)} \right)^2 \in (0, 1).$$

And so

$$\frac{K^2}{1 - \delta} = (u'_0(x_0))^2 < w^2(t), \quad \text{i.e.} \quad K^2 < (1 - \delta)w^2(t).$$

Therefore

$$\frac{d}{dt}w(t) \leq -\frac{1}{2}w^2(t) [1 - (1 - \delta)] = -\delta w^2(t), \quad t \in [0, T),$$

which leads to

$$w(t) \leq \frac{u'_0(x_0)}{1 + \delta t u'_0(x_0)} \rightarrow -\infty, \quad \text{as} \quad t \rightarrow -\frac{1}{\delta u'_0(x_0)}.$$

This implies

$$T \leq -\frac{1}{\delta u'_0(x_0)} = \frac{\inf_{x \in \mathbb{S}} u'_0(x)}{K^2 - (\inf_{x \in \mathbb{S}} u'_0(x))^2} < +\infty.$$

In consequence, we have

$$\liminf_{t \uparrow T} \left(\inf_{x \in \mathbb{S}} u_x(t, x) \right) = -\infty.$$

This completes the proof of Theorem 3.2. \square

Remark 3.1. We can apply Lemma 2.4 to verify the above theorem under the same conditions. In fact, if we define $w(t) = u_x(t, \xi(t)) = \inf_{x \in \mathbb{S}} [u_x(t, x)]$, then for all $t \in [0, T)$,

$u_{xx}(t, \xi(t)) = 0$. Thus if $(\sqrt{3}/\pi)|\mu_0 + \kappa| \geq \mu_1$, one finds that

$$\frac{d}{dt}w(t) \leq -\frac{1}{2}w^2(t) + \frac{1}{2}K^2,$$

where K is the same as Theorem 3.2. Then by means of the assumptions of Theorem 3.2 and following the line of the proof of Theorem 3.2, we see that if

$$w(0) < -\sqrt{2\mu_1 \left(\frac{\sqrt{3}}{3}|\mu_0 + \kappa| - \frac{1}{2}\mu_1 \right)},$$

then T is finite and $\liminf_{t \uparrow T} \left(\inf_{x \in \mathbb{S}} u_x(t, x) \right) = -\infty$.

Recall the definition of the extrema $m_1(t)$, $m_2(t)$ in (2.11) of the slope $u_x(t, x)$ on the circle \mathbb{S} , we may get the following wave-breaking result.

Theorem 3.3. *Let $u_0 \in H^s(\mathbb{S})$, $s > 3/2$ and $T > 0$ be the maximal time of existence of the corresponding solution $u(t, x)$ to (1.2) with the initial data u_0 . If*

$$m_1(0) + m_2(0) < -8|\kappa| \quad \text{when} \quad \frac{2\sqrt{3}}{3}|\mu_0| < \mu_1, \quad \text{or}$$

$$m_1(0) + m_2(0) < -8|\kappa| - 2\sqrt{2}C_1 \quad \text{when} \quad \frac{2\sqrt{3}}{3}|\mu_0| \geq \mu_1$$

with $C_1 := \sqrt{\left|\frac{\sqrt{3}}{3}|\mu_0| - \frac{1}{2}\mu_1\right| \mu_1}$, then the corresponding solution $u(t, x)$ to (1.2) blows up in finite time T .

Proof. As discussed above, it suffices to consider the case $s = 3$. In view of (3.1), (2.1), (2.2) and (1.7), together with Remark 1.1 applied, we have

$$\begin{aligned} u_{tx} &= -u_x^2 - uu_{xx} - A^{-1}\partial_x^2(2u\mu_0 + \frac{1}{2}u_x^2) - 2\kappa A^{-1}\partial_x^2 u \\ &= -\frac{1}{2}u_x^2 - uu_{xx} + 2\mu_0(u - \mu_0) - \frac{1}{2}\mu_1^2 - 2\kappa A^{-1}\partial_x^2 u. \end{aligned} \quad (3.8)$$

Thanks to (1.6), we obtain that

$$\begin{aligned} A^{-1}\partial_x^2 u &= \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12}\right)\mu(\partial_x^2 u) + \left(x - \frac{1}{2}\right)\int_0^1 \int_0^y \partial_x^2 u(s) ds dy \\ &\quad + \int_0^1 \int_x^y \int_0^s \partial_x^2 u(r) dr ds dy, \end{aligned}$$

which implies

$$\begin{aligned} |A^{-1}\partial_x^2 u| &= \left| \frac{2x-1}{2} \int_0^1 \int_0^y \partial_x^2 u(s) ds dy + \int_0^1 \int_x^y \int_0^s \partial_x^2 u(r) dr ds dy \right| \\ &= \left| \left(x - \frac{1}{2}\right) \int_0^1 (\partial_x u(y) - \partial_x u(0)) dy + \int_0^1 \int_x^y (\partial_x u(s) - \partial_x u(0)) ds dy \right| \\ &\leq (m_2 - m_1) \left(\left|x - \frac{1}{2}\right| + \int_0^1 |y - x| dy \right) \\ &\leq (m_2 - m_1) \left(\left|x - \frac{1}{2}\right| + x^2 - x + \frac{1}{2} \right) \leq m_2 - m_1. \end{aligned}$$

From this, together with (3.8), (2.4), the fact $u_{xx}(t, x_i(t)) = 0$ for a.e. $t \in [0, T]$, and Lemma 2.4 applied, we deduce that

$$\frac{d}{dt}m_i \leq -\frac{1}{2}m_i^2 + \frac{\sqrt{3}}{3}|\mu_0|\mu_1 - \frac{1}{2}\mu_1^2 + 2|\kappa|(m_2 - m_1), \quad i = 1, 2. \quad (3.9)$$

Summing up the above two inequalities gives

$$\frac{d}{dt}(m_1 + m_2) \leq -\frac{1}{2}(m_1^2 + m_2^2) + \mu_1 \left(\frac{2\sqrt{3}}{3}|\mu_0| - \mu_1 \right) + 4|\kappa|(m_2 + m_1) - 8|\kappa|m_1.$$

If $\frac{2\sqrt{3}}{3}|\mu_0| < \mu_1$, one has

$$\frac{d}{dt}(m_1 + m_2) \leq -\frac{1}{2}(m_1^2 + m_2^2) + 4|\kappa|(m_2 + m_1) - 8|\kappa|m_1 - 2C_1^2. \quad (3.10)$$

Since $(m_1 + m_2)(0) < -8|\kappa|$, there is $\delta_0 \in (0, \frac{1}{2}]$ such that $(m_1 + m_2)(0) \leq -\alpha$ with $\alpha = 8|\kappa| + \delta_0 > 8|\kappa|$.

We first claim that there holds

$$(m_1 + m_2)(t) \leq -\alpha \quad \text{for } \forall \quad t \in (0, T).$$

Indeed, note that $\bar{m}(t) := (m_1 + m_2)(t) + \alpha$ is continuous on $[0, T]$. If the above inequality does not hold, we can find a $t_0 \in (0, T)$ such that $\bar{m}(t_0) > 0$. Denote

$$t_1 = \max\{t < t_0 \mid \bar{m}(t_0) = 0\}.$$

Then

$$\bar{m}(t_1) = 0 \quad \text{and} \quad \frac{d}{dt}\bar{m}(t_1) \geq 0. \quad (3.11)$$

While thanks to

$$m_1(t_1) \leq \frac{1}{2}\bar{m}(t_1) - \frac{\alpha}{2} = -\frac{\alpha}{2},$$

we get from (3.10) that

$$\begin{aligned} \frac{d}{dt}\bar{m}(t_1) &= \frac{d}{dt}(m_1 + m_2)(t_1) < -\frac{1}{2}(m_1^2 + m_2^2)(t_1) + 4|\kappa|(m_2 + m_1)(t_1) - 8|\kappa|m_1(t_1) \\ &\leq -\frac{1}{2}m_1^2(t_1) - 4|\kappa|\alpha - 8|\kappa|m_1(t_1) \\ &= -\frac{1}{2}(m_1(t_1) + 8|\kappa|)^2 - 4|\kappa|(\alpha - 8|\kappa|) \leq 0. \end{aligned}$$

This yields a contradiction with (3.11), and this completes the proof of the claim.

Putting the obtained estimate $m_1(t) \leq \frac{m_1(t) + m_2(t)}{2} \leq -\frac{\alpha}{2} < -4|\kappa|$ back into (3.9) with $i = 1$, we find

$$\begin{aligned} \frac{d}{dt}(m_1(t) + 4|\kappa|) &= \frac{d}{dt}m_1(t) \leq -\frac{1}{2}m_1^2 - C_1^2 + 2|\kappa|(m_2 + m_1) - 4|\kappa|m_1 \\ &\leq -\frac{1}{2}m_1^2 - C_1^2 - 2|\kappa|\alpha - 4|\kappa|m_1 \\ &\leq -\frac{1}{2}(m_1 + 4|\kappa|)^2 - 2|\kappa|(\alpha - 4|\kappa|) - C_1^2 \\ &< -\frac{1}{2}(m_1 + 4|\kappa|)^2 \quad \text{for a.e. } t \in (0, T), \end{aligned} \quad (3.12)$$

which implies $m_1(t) + 4|\kappa| < 0$ on $(0, T)$. From this and the fact that $m_1(t) + 4|\kappa|$ is locally Lipschitz on $(0, T)$, we see that $\frac{1}{m_1(t) + 4|\kappa|}$ is also Lipschitz on $(0, T)$. Being locally Lipschitz, the $\frac{1}{m_1(t) + 4|\kappa|}$ is absolutely continuous on $(0, T)$, it is then inferred from (3.12) that

$$\frac{d}{dt} \left(\frac{1}{m_1(t) + 4|\kappa|} \right) \geq \frac{1}{2} \quad \text{for a.e. } t \in (0, T).$$

Therefore, we get

$$m_1(t) \leq \frac{2(m_1(0) + 4|\kappa|)}{2 + (m_1(0) + 4|\kappa|)t} - 4|\kappa| \quad \text{for a.e. } t \in (0, T),$$

which implies that the life-span $T \leq \frac{-2}{m_1(0) + 4|\kappa|}$.

On the other hand, if $\frac{2\sqrt{3}}{3}|\mu_0| \geq \mu_1$, we find from (3.9) that

$$\frac{d}{dt}m_i \leq -\frac{1}{2}m_i^2 + 2|\kappa|(m_2 - m_1) + C_1^2, \quad i = 1, 2. \quad (3.13)$$

Summing up the above two inequalities gives

$$\frac{d}{dt}(m_1 + m_2) \leq -\frac{1}{2}(m_1^2 + m_2^2) + 4|\kappa|(m_2 + m_1) - 8|\kappa|m_1 + 2C_1^2. \quad (3.14)$$

Since $(m_1 + m_2)(0) < -8|\kappa| - 2\sqrt{2}C_1$, then there is $\delta_0 \in (0, \frac{1}{2}]$ such that $(m_1 + m_2)(0) \leq -\alpha - 2\sqrt{2}(1 + \delta_0)C_1$ with $\alpha = 8|\kappa| + \delta_0 > 8|\kappa|$.

Again we first claim that there holds for all $t \in (0, T)$

$$(m_1 + m_2)(t) \leq -\alpha - 2\sqrt{2}(1 + \delta_0)C_1.$$

Indeed, similar to the argument above, note that $\bar{m}(t) := (m_1 + m_2)(t) + \alpha + 2\sqrt{2}(1 + \delta_0)C_1$ is continuous on $[0, T)$. If the above inequality does not hold, we can find a $t_0 \in (0, T)$ such that $\bar{m}(t_0) \geq 0$. Denote

$$t_1 = \max\{t < t_0 \mid \bar{m}(t_0) = 0\}.$$

Then

$$\bar{m}(t_1) = 0 \quad \text{and} \quad \frac{d}{dt}\bar{m}(t_1) \geq 0. \quad (3.15)$$

While thanks to

$$m_1(t_1) \leq \frac{1}{2}\bar{m}(t_1) - \frac{\alpha}{2} - \sqrt{2}(1 + \delta_0)C_1 = -\frac{\alpha}{2} - \sqrt{2}(1 + \delta_0)C_1$$

and

$$m_2(t_1) = \bar{m}(t_1) - \alpha - 2\sqrt{2}(1 + \delta_0)C_1 - m_1(t_1) = -\alpha - 2\sqrt{2}(1 + \delta_0)C_1 - m_1(t_1),$$

we get from (3.14) that

$$\begin{aligned} \frac{d}{dt}\bar{m}(t_1) &\leq -\frac{1}{2}(m_1^2 + m_2^2)(t_1) + 4|\kappa|(m_2 + m_1)(t_1) - 8|\kappa|m_1(t_1) + 2C_1^2 \\ &= -\frac{1}{2}m_1^2(t_1) - \frac{1}{2}\left(m_1(t_1) + \alpha + 2\sqrt{2}(1 + \delta_0)C_1\right)^2 \\ &\quad - 4|\kappa|\left(\alpha + 2\sqrt{2}(1 + \delta_0)C_1\right) - 8|\kappa|m_1(t_1) + 2C_1^2 \\ &= -\frac{1}{4}\left(2m_1(t_1) + \alpha + 2\sqrt{2}(1 + \delta_0)C_1 + 8|\kappa|\right)^2 + \frac{1}{4}\left(\alpha + 2\sqrt{2}(1 + \delta_0)C_1 + 8|\kappa|\right)^2 \\ &\quad - \frac{1}{2}\left(\alpha + 2\sqrt{2}(1 + \delta_0)C_1\right)^2 + 2C_1^2 - 4|\kappa|\left(\alpha + 2\sqrt{2}(1 + \delta_0)C_1\right), \end{aligned}$$

which together with the fact $\alpha > 8|\kappa|$ implies

$$\begin{aligned} \frac{d}{dt}\bar{m}(t_1) &\leq \frac{1}{4}\left(\alpha + 2\sqrt{2}(1 + \delta_0)C_1 + 8|\kappa|\right)^2 - \frac{1}{2}\left(\alpha + 2\sqrt{2}(1 + \delta_0)C_1\right)^2 \\ &\quad + 2C_1^2 - 4|\kappa|\left(\alpha + 2\sqrt{2}(1 + \delta_0)C_1\right) \\ &= -\frac{1}{4}\left(\alpha + 2\sqrt{2}(1 + \delta_0)C_1\right)^2 + 2C_1^2 + 16|\kappa|^2 < 0. \end{aligned}$$

This yields a contradiction with (3.15), and the proof of the claim is complete.

Therefore, $m_1(t) \leq \frac{m_1(t) + m_2(t)}{2} \leq -\frac{\alpha}{2} - \sqrt{2}(1 + \delta_0)C_1 < -4|\kappa| - \sqrt{2}(1 + \delta_0)C_1$ back into (3.13) with $i = 1$, we find for all $t \in (0, T)$

$$\begin{aligned} \frac{d}{dt}(m_1(t) + 4|\kappa|) &= \frac{d}{dt}m_1(t) \leq -\frac{1}{2}m_1^2 + C_1^2 + 2|\kappa|(m_2 + m_1) - 4|\kappa|m_1 \\ &\leq -\frac{1}{2}m_1^2 + C_1^2 - 2|\kappa|\left(\alpha + 2\sqrt{2}(1 + \delta_0)C_1\right) - 4|\kappa|m_1 \\ &= -\frac{1}{2}(m_1 + 4|\kappa|)^2 + C_1^2 - 2|\kappa|\left(\alpha - 4|\kappa| + 2\sqrt{2}(1 + \delta_0)C_1\right) \\ &\leq -\frac{\delta_0}{2(1 + \delta_0)}(m_1 + 4|\kappa|)^2, \end{aligned}$$

which implies $m_1(t) < -4|\kappa| - \sqrt{2}(1+\delta_0)C_1$ on $(0, T)$. From this and the fact that $m_1(t)$ is locally Lipschitz on $(0, T)$, we see that $\frac{1}{m_1(t)+4|\kappa|}$ is also Lipschitz on $(0, T)$. Being locally Lipschitz, the $\frac{1}{m_1(t)+4|\kappa|}$ is absolutely continuous on $(0, T)$, it is then inferred from (3.12) that

$$\frac{d}{dt} \left(\frac{1}{m_1(t) + 4|\kappa|} \right) \geq \frac{\delta_0}{2(1+\delta_0)} \quad \text{for a.e. } t \in (0, T).$$

Therefore, we again get

$$m_1(t) \leq \frac{2(1+\delta_0)(m_1(0) + 4|\kappa|)}{2(1+\delta_0) + \delta_0(m_1(0) + 4|\kappa|)t} - 4|\kappa| \quad \text{for a.e. } t \in (0, T),$$

which implies that the life-span $T \leq -\frac{2(1+\delta_0)}{\delta_0(m_1(0) + 4|\kappa|)}$. This completes the proof of Theorem 3.3. \square

Remark 3.2. *Theorem 3.3 does not overlap with Theorems 3.1, or Theorem 3.2, which may be easily verified when we consider the two cases, $\mu_1 \gg |\kappa| \sim |\mu_0|$ and $\mu_1 \ll |\kappa| \sim |\mu_0|$ respectively.*

Using the conserved quantities H_2 , we can derive the following wave-breaking result.

Theorem 3.4. *Let $u_0 \in H^s(\mathbb{S})$, $s > 3/2$ and $T > 0$ be the maximal time of existence of the corresponding solution $u(t, x)$ to (1.2) with the initial data u_0 . If*

$$(\mu_0 + \kappa)H_2 < \frac{1}{8}\mu_1^4 + \frac{1}{2}\mu_0(\mu_0 + \kappa)(2\mu_0^2 + \mu_1^2), \quad \mu_0(\mu_0 + \kappa) \geq 0, \quad \text{or} \quad (3.16)$$

$$(\mu_0 + \kappa)H_2 < \frac{1}{8}\mu_1^4 + \frac{1}{2}\mu_0(\mu_0 + \kappa) \left(2\mu_0^2 + \left(1 + \frac{1}{2\pi^2}\right)\mu_1^2 \right), \quad \mu_0(\mu_0 + \kappa) < 0, \quad (3.17)$$

where μ_0, μ_1 are defined in (2.1) and (2.2), then the corresponding solution $u(t, x)$ to (1.2) blows up in finite time T with

$$0 < T \leq 6 + \frac{1 + \left| \int_{\mathbb{S}} u_{0x}^3(x) dx \right|}{\frac{3}{2}\mu_1^4 + 6\mu_0(\mu_0 + \kappa)(\mu_1^2 + 2\mu_0^2) - 12(\mu_0 + \kappa)H_2}, \quad \text{if } \mu_0(\mu_0 + \kappa) \geq 0$$

or

$$0 < T \leq 6 + \frac{1 + \left| \int_{\mathbb{S}} u_{0x}^3(x) dx \right|}{\frac{3}{2}\mu_1^4 + \mu_0(\mu_0 + \kappa) \left(\left(6 + \frac{3}{\pi^2}\right)\mu_1^2 + 12\mu_0^2 \right) - 12(\mu_0 + \kappa)H_2}, \quad \text{if } \mu_0(\mu_0 + \kappa) < 0$$

such that

$$\liminf_{t \uparrow T} \left(\inf_{x \in \mathbb{S}} u_x(t, x) \right) = -\infty.$$

Proof. Again it suffices to consider the case $s = 3$. Recall that

$$H_2 = \int_{\mathbb{S}} \left(\mu_0 u^2 + \kappa u^2 + \frac{1}{2} u u_x^2 \right) dx$$

is independent of time t . In view of (3.3), we obtain

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{S}} u_x^3 dx &= -\frac{1}{2} \int_{\mathbb{S}} u_x^4 dx - \frac{3}{2} \mu_1^4 + 6(\mu_0 + \kappa) \int_{\mathbb{S}} u u_x^2 dx \\
&\quad - 6(\mu_0 + \kappa) \mu_0 \int_{\mathbb{S}} u_x^2 dx \\
&= -\frac{1}{2} \int_{\mathbb{S}} u_x^4 dx - \frac{3}{2} \mu_1^4 + 12(\mu_0 + \kappa) H_2 - 6\mu_0(\mu_0 + \kappa) \mu_1^2 \\
&\quad - 12\mu_0(\mu_0 + \kappa) \int_{\mathbb{S}} u^2 dx.
\end{aligned} \tag{3.18}$$

If $\mu_0(\mu_0 + \kappa) \geq 0$, it then follows from Hölder's inequality that

$$\mu_0(\mu_0 + \kappa) \int_{\mathbb{S}} u^2 dx \geq \mu_0(\mu_0 + \kappa) \left(\int_{\mathbb{S}} u dx \right)^2 = \mu_0^3(\mu_0 + \kappa).$$

Hence, we have

$$\begin{aligned}
&\frac{3}{2} \mu_1^4 - 12(\mu_0 + \kappa) H_2 + 6\mu_0(\mu_0 + \kappa) \mu_1^2 + 12\mu_0(\mu_0 + \kappa) \int_{\mathbb{S}} u^2 dx \\
&\geq \frac{3}{2} \mu_1^4 + 6\mu_0(\mu_0 + \kappa)(\mu_1^2 + 2\mu_0^2) - 12(\mu_0 + \kappa) H_2 =: C_0
\end{aligned} \tag{3.19}$$

Thanks to the assumption (3.16), we get $C_0 > 0$.

On the other hand, if $\mu_0(\mu_0 + \kappa) < 0$, we get from (2.7) that

$$\mu_0(\mu_0 + \kappa) \int_{\mathbb{S}} u^2 dx \geq \mu_0(\mu_0 + \kappa) \left(\frac{1}{4\pi^2} \mu_1^2 + \mu_0^2 \right).$$

It then follows that

$$\begin{aligned}
&\frac{3}{2} \mu_1^4 - 12(\mu_0 + \kappa) H_2 + 6\mu_0(\mu_0 + \kappa) \mu_1^2 + 12\mu_0(\mu_0 + \kappa) \int_{\mathbb{S}} u^2 dx \\
&\geq \frac{3}{2} \mu_1^4 + \mu_0(\mu_0 + \kappa) \left(\left(6 + \frac{3}{\pi^2}\right) \mu_1^2 + 12\mu_0^2 \right) - 12(\mu_0 + \kappa) H_2 =: C_0
\end{aligned} \tag{3.20}$$

Thanks to the assumption (3.16), we also get $C_0 > 0$.

In view of (3.18)-(3.20), together with Hölder's inequality applied, we deduce that

$$\frac{d}{dt} \int_{\mathbb{S}} u_x^3 dx \leq -\frac{1}{2} \int_{\mathbb{S}} u_x^4 dx - C_0 \leq -\frac{1}{2} \left(\int_{\mathbb{S}} u_x^3 dx \right)^{\frac{4}{3}} - C_0.$$

Define $V(t) = \int_{\mathbb{S}} u_x^3(t, x) dx$ with $t \in [0, T)$. It is clear that

$$\frac{d}{dt} V(t) \leq -\frac{1}{2} (V(t))^{\frac{4}{3}} - C_0 \leq -C_0 < 0, \quad t \in [0, T).$$

Let $t_1 = (1 + |V(0)|)/C_0$. Then following the proof of Theorem 3.1, we have

$$T \leq t_1 + 6 < +\infty.$$

This implies the desired result as in Theorem 3.4. \square

4. EXISTENCE OF GLOBAL STRONG SOLUTION

In this section, attention is now turned to specifying conditions under which the local strong solution to the initial-value problem (1.1) can be extended to a global one.

Theorem 4.1. *If the initial potential $m_0 \in H^1(\mathbb{S})$ satisfies that $m_0 + \kappa$ does not change the sign, then the solution $u(t)$ to the initial-value problem (1.1) exists permanently in time.*

Proof. Let T be the maximal time of existence of the solution u to (1.2) with the initial data u_0 , guaranteed by Proposition 2.1.

Assume $m_0 + \kappa \geq 0$. We prove that the solution $u(t, x)$ exists globally in time. Indeed, thanks to Lemma 2.3 and Remark 2.3, we find $m(t) + \kappa \geq 0$ on $[0, T) \times \mathbb{S}$. Given $t \in [0, T)$, by the periodicity in the x -variable, there is a $\xi(t) \in (0, 1)$ such that $u_x(t, \xi(t)) = 0$. Therefore, for $x \in [\xi(t), \xi(t) + 1]$ we have

$$-u_x(t, x) = -\int_{\xi(t)}^x \partial_x^2 u(t, x) dx = \int_{\xi(t)}^x (m(t, x) + \kappa) dx - \int_{\xi(t)}^x [\mu(u) + \kappa] dx,$$

which leads to

$$\begin{aligned} -u_x(t, x) &\leq \int_{\xi(t)}^{\xi(t)+1} (m(t, x) + \kappa) dx - (\mu_0 + \kappa)(x - \xi(t)) \\ &= \int_{\mathbb{S}} (m_0 + \kappa) dx - (\mu_0 + \kappa)(x - \xi(t)) = (\mu_0 + \kappa)(1 - x + \xi(t)) \leq |\mu_0 + \kappa|. \end{aligned} \quad (4.1)$$

On the other hand, if $m_0 + \kappa \leq 0$, then $m(t) + \kappa \leq 0$ on $[0, T) \times \mathbb{S}$. Using the same notation as above, we find that

$$\begin{aligned} -u_x(t, x) &= -\int_{\xi(t)}^x \partial_x^2 u(t, x) dx = \int_{\xi(t)}^x [m(t, x) + \kappa] dx - \int_{\xi(t)}^x [\mu(u) + \kappa] dx \\ &\leq -(\mu_0 + \kappa)(x - \xi(t)) \leq |\mu_0 + \kappa|. \end{aligned} \quad (4.2)$$

From (4.1) and (4.2), we deduce that u exists permanently as a consequence of Proposition 3.1. \square

Theorem 4.2. *If the initial profile $u_0 \in H^3(\mathbb{S})$ is such that*

$$\|\partial_x^3 u_0\|_{L^2} \leq 2\sqrt{3}|\mu_0 + \kappa|, \quad (4.3)$$

then the initial-value problem (1.2) admits global solutions in time.

Proof. Let T be the maximal time of existence of the solution u to (1.2) with the initial data u_0 , given by Proposition 2.1.

By Lemma 2.1, we get

$$\max(\partial_x^2 u_0)^2 \leq \frac{1}{12} \int_{\mathbb{S}} (\partial_x^3 u_0)^2 dx,$$

which gives rise to

$$\|\partial_x^2 u_0\|_{L^\infty} \leq \frac{\sqrt{3}}{6} \|\partial_x^3 u_0\|_{L^2}. \quad (4.4)$$

If $\mu_0 + \kappa \geq 0$, it then is inferred from (4.4) and the assumption (4.3) that

$$m_0 + \kappa = \mu_0 + \kappa - \partial_x^2 u_0 \geq \mu_0 + \kappa - \frac{\sqrt{3}}{6} \|\partial_x^3 u_0\|_{L^2} \geq 0$$

Similarly, if $\mu_0 + \kappa \leq 0$, one obtains from (4.4) and (4.3) that

$$m_0 + \kappa = \mu_0 + \kappa - \partial_x^2 u_0 \leq \mu_0 + \kappa + \frac{\sqrt{3}}{6} \|\partial_x^3 u_0\|_{L^2} \leq 0.$$

Therefore, in view of Theorem 4.1, the proof of this theorem is complete. \square

5. EXISTENCE OF GLOBAL WEAK SOLUTION

In this section, we establish the existence of an admissible global weak solution to (1.2), which may be stated as follows.

Theorem 5.1. *Assume that $u_0 \in H^1(\mathbb{S})$. Then the initial-value problem (1.1) has an admissible global weak solution, $u = u(t, x)$, in the sense of Definition 1.2. Furthermore, this weak solution $u(t, x)$ satisfies the following properties.*

(i) *One-sided supernorm estimate: There exists a positive constant $C = C(u_0)$ such that the following one-sided L^∞ norm estimate on the first-order spatial derivative holds in the sense of distribution:*

$$\partial_x u(t, x) \leq \frac{1}{t} + C, \quad \forall \quad t > 0, x \in \mathbb{S}. \quad (5.1)$$

(ii) *Space-time higher integrability estimate.*

$$\partial_x u \in L_{loc}^p(\mathbb{R}^+ \times \mathbb{S}), \quad \forall 1 \leq p < 3,$$

i.e., for any $0 < T < +\infty$, there exists a positive constant $C_1 = C_1(T, p)$ such that

$$\int_0^T \int_{\mathbb{S}} |\partial_x u_\varepsilon(t, x)|^p dx dt \leq C_1, \quad \forall 1 \leq p < 3. \quad (5.2)$$

The proof of this theorem is motivated by the one of Theorem 1.2 in [41]. This method, as far as we know, was first used by Zhang and Zheng to study an admissible global solution to a variational wave equation in [45].

5.1. Viscous Approximate Solutions. We obtain the existence of a global weak solution to the initial-value problem (1.2) by proving compactness of a sequence of smooth functions $\{u_\varepsilon\}_{\varepsilon>0}$ solving the following viscous problems with the initial data $u_{\varepsilon 0}(x) = \phi_\varepsilon * u_0$,

$$\begin{cases} \partial_t u_\varepsilon + u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon - \varepsilon \partial_x^2 u_\varepsilon = 0, & t > 0, \quad x \in \mathbb{R}, \\ (\mu - \partial_x^2) P_\varepsilon = 2\mu(u_\varepsilon)u_\varepsilon + \frac{1}{2}(\partial_x u_\varepsilon)^2 + 2\kappa u_\varepsilon, & t > 0, \quad x \in \mathbb{R}, \\ u_\varepsilon(t, x+1) = u_\varepsilon(t, x), & t \geq 0, \quad x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon 0}(x), & x \in \mathbb{R}, \end{cases} \quad (5.3)$$

or equivalently,

$$\begin{cases} \partial_t m_\varepsilon - \varepsilon \partial_x^2 m_\varepsilon + 2\kappa \partial_x u_\varepsilon + u_\varepsilon \partial_x m_\varepsilon + 2m_\varepsilon \partial_x u_\varepsilon = 0, & t > 0, \quad x \in \mathbb{R}, \\ m_\varepsilon = (\mu - \partial_x^2)u_\varepsilon, & t \geq 0, \quad x \in \mathbb{R}, \\ u_\varepsilon(t, x+1) = u_\varepsilon(t, x), & t \geq 0, \quad x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon 0}(x), & x \in \mathbb{R}, \end{cases} \quad (5.4)$$

where the truncating family $\{\phi_\varepsilon(x)\}_{\varepsilon>0}$ satisfies

$$\phi_\varepsilon(x) = \varepsilon^{-1} \phi(x/\varepsilon) \quad \text{with} \quad \varepsilon > 0, \quad \phi \in C_c^\infty(\mathbb{R}), \quad \phi \geq 0, \quad \|\phi\|_{L^1} = 1. \quad (5.5)$$

The existence, uniqueness, and basic energy estimate on this approximate solution sequence are given in the following proposition.

Proposition 5.1. *Let $\varepsilon > 0$ and $u_{0\varepsilon} \in H^k(\mathbb{S})$ for some $k \geq 1$. Then there exists a unique solution $u_\varepsilon \in C(\mathbb{R}^+; H^k(\mathbb{S}))$ to the initial-value problem (5.3). Furthermore, the following energy identities hold for all $t \geq 0$.*

$$\mu(u_\varepsilon(t)) = \mu(u_{0\varepsilon}) \quad \text{and} \quad \int_{\mathbb{S}} (\partial_x u_\varepsilon)^2(t, x) dx + 2\varepsilon \int_{\mathbb{S}} (\partial_x^2 u_\varepsilon)^2(t, x) dx = \int_{\mathbb{S}} (\partial_x u_{0\varepsilon})^2 dx. \quad (5.6)$$

Remark 5.1. *Thanks to (5.5), together with Young's inequality applied, we deduce that*

$$\begin{aligned} \mu(u_{0\varepsilon}) &= \int_{\mathbb{S}} \int_{\mathbb{R}} \frac{1}{\varepsilon} \phi\left(\frac{y}{\varepsilon}\right) u_0(x-y) dy dx = \int_{\mathbb{R}} \frac{1}{\varepsilon} \phi\left(\frac{y}{\varepsilon}\right) \left(\int_{\mathbb{S}} u_0(x-y) dx \right) dy \\ &= \mu(u_0) \int_{\mathbb{R}} \frac{1}{\varepsilon} \phi\left(\frac{y}{\varepsilon}\right) dy = \mu(u_0) = \mu_0 \end{aligned}$$

and

$$\int_{\mathbb{S}} (\partial_x u_{0\varepsilon})^2 dx = \|\phi_\varepsilon * \partial_x u_0\|_{L^2}^2 \leq \|\phi_\varepsilon\|_{L^1} \|\partial_x u_0\|_{L^2}^2 = \|\partial_x u_0\|_{L^2}^2 = \mu_1^2. \quad (5.7)$$

The strategy of the proof of Proposition 5.1 is rather routine. For the sake of simplicity, we will only sketch the necessary estimates. While for the convenience of presentation, we will omit the subscript ε in u_ε in the following proof.

Proof of Proposition 5.1. First, following the standard argument for a nonlinear parabolic equation, one can obtain the local well-posedness result that for $u_{0\varepsilon} \in H^k(\mathbb{S})$, there exists a positive constant T_0 such that (5.3) has a unique solution

$$u = u(t, x) \in C([0, T_0], H^k(\mathbb{S})) \cap L^2([0, T_0], H^{k+1}(\mathbb{S})).$$

We denote the life span of the solution $u(t, x)$ by T . Then, (5.6) holds for all $0 \leq t < T$.

Next we claim that if the life span $T < +\infty$, i.e., $u \in C([0, T], H^k(\mathbb{S}))$, and

$$\lim_{t \rightarrow T} \|u(t, \cdot)\|_{H^k(\mathbb{S})} = +\infty, \quad T < +\infty, \quad (5.8)$$

then

$$\lim_{t \rightarrow T} \int_0^t \|\partial_x u(\tau, \cdot)\|_{L^\infty(\mathbb{S})} d\tau = +\infty, \quad T < +\infty.$$

Indeed, assume that the maximal existence time $T < +\infty$. It then follows from the equation in (5.3), together with (5.6), that for $t < T$

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \|u\|_{H^k(\mathbb{S})}^2 + \varepsilon \|\partial_x u\|_{H^k(\mathbb{S})}^2 \\ &= \sum_{\alpha=0}^k \int_{\mathbb{S}} \left(\frac{1}{2} \partial_x u (\partial_x^\alpha u)^2 + (u \partial_x^{1+\alpha} u - \partial_x^\alpha (u \partial_x u)) \partial_x^\alpha u - \partial_x^{1+\alpha} P \partial_x^\alpha u \right) dx. \end{aligned} \quad (5.9)$$

Note that

$$\sum_{\alpha=0}^k \int_{\mathbb{S}} \left(\frac{1}{2} \partial_x u (\partial_x^\alpha u)^2 \right) dx \leq C_k \|\partial_x u(t)\|_{L^\infty(\mathbb{S})} \|u(t)\|_{H^k(\mathbb{S})}^2 \quad (5.10)$$

and

$$\left| \int_{\mathbb{S}} \partial_x^{1+\alpha} P \partial_x^\alpha u dx \right| \leq \|\partial_x^\alpha u(t)\|_{L^2(\mathbb{S})} \|\partial_x^{1+\alpha} P\|_{L^2(\mathbb{S})}. \quad (5.11)$$

Due to Remark 1.1, we apply a standard elliptic regularity estimate to (1.8) to obtain that for $0 \leq \alpha \leq k$

$$\begin{aligned} \|\partial_x^{1+\alpha} P\|_{L^2} &\leq C(|\mu(u)|^2 + \|u\|_{H^1}^2 + \kappa\|u\|_{L^2(\mathbb{S})} + \|\partial_x^{\alpha-1}((\partial_x u)^2)\|_{L^2}^2) \\ &\leq C(|\mu(u)|^2 + \|u\|_{H^1}^2 + \kappa\|u\|_{L^2(\mathbb{S})} + \|\partial_x u\|_{L^\infty}\|u\|_{H^\alpha}). \end{aligned} \quad (5.12)$$

Applying the Kato-Ponce commutator estimate [29] yields

$$\|u\partial_x^{1+\alpha} u - \partial_x^\alpha(u\partial_x u)\|_{L^2(\mathbb{S})} \leq C_k \|\partial_x u(t)\|_{L^\infty(\mathbb{S})} \|\partial_x^\alpha u(t)\|_{L^2(\mathbb{S})}, \quad (5.13)$$

which, together with (5.10), (5.11), (5.12) and (5.6) applied to (5.9), leads to

$$\frac{d}{dt} \|u\|_{H^k(\mathbb{S})}^2 + 2\varepsilon \|\partial_x u\|_{H^k(\mathbb{S})}^2 \leq C_k (\|\partial_x u\|_{L^\infty} + 1) \|u\|_{H^k(\mathbb{S})}^2. \quad (5.14)$$

Hence, if $\lim_{t \rightarrow T} \int_0^t \|\partial_x u(\tau, \cdot)\|_{L^\infty(\mathbb{S})} d\tau < +\infty$, then applying Gronwall's inequality to (5.14), we get $\lim_{t \rightarrow T} \|u(t, \cdot)\|_{H^k(\mathbb{S})} < +\infty$, which contradicts (5.8). This completes the proof of the claim.

On the other hand, thanks to Lemma 2.1, we get

$$\max_{x \in \mathbb{S}} (\partial_x u(t, x))^2 \leq \frac{1}{12} \int_{\mathbb{S}} (\partial_x^2 u)^2(t, x) dx.$$

From this, together with (5.6), (5.7) and Hölder's inequality, we obtain that for any $0 \leq t < T$

$$\int_0^t \|\partial_x u(\tau, \cdot)\|_{L^\infty(\mathbb{S})} d\tau \leq \frac{\sqrt{3}}{6} \int_0^t \|\partial_x^2 u(\tau, \cdot)\|_{L^2(\mathbb{S})} d\tau \leq \frac{\sqrt{6}}{12\sqrt{\varepsilon}} T^{\frac{1}{2}} \|\partial_x u_0\|_{L^2(\mathbb{S})},$$

which implies that the life-span $T = +\infty$. Furthermore, (5.6) now holds on $[0, +\infty)$. This completes the proof of Proposition 5.1. \square

5.2. Uniform A Priori Estimates. Let $u_0 \in H^1(\mathbb{S})$ and $u_\varepsilon(t, x)$ be the unique global-in-time solution to (5.3) obtained in Proposition 3.1 which satisfies the energy identity (5.6). To obtain the compactness of this approximate solution sequence, we need some *a priori* estimates in addition to (5.6). In this subsection, we derive the uniform one-sided supernorm estimate (5.1) and the space-time higher integrability estimates (5.2) on $\partial_x u_\varepsilon(t, x)$, which are essential for our compactness argument.

We start with the uniform one-sided supernorm estimate, which is similar to Oleinik's entropy condition for the theory of shock waves [40].

Proposition 5.2. *There holds*

$$\partial_x u_\varepsilon(t, x) \leq \frac{2}{t} + L_0, \quad \forall t > 0, \quad x \in \mathbb{S} \quad (5.15)$$

with the constant

$$L_0 := \sqrt{2(\mu_0 + \kappa)^2 + \frac{7}{6}\mu_1^2}.$$

Proof. Set $q_\varepsilon = \partial_x u_\varepsilon$. Differentiating the first equation in (5.3) with respect to x , we get from Proposition 5.1 and Remark 5.1 that

$$\begin{cases} \partial_t q_\varepsilon + u_\varepsilon \partial_x q_\varepsilon - \varepsilon \partial_x^2 q_\varepsilon + \frac{1}{2}(q_\varepsilon)^2 = 2(u_\varepsilon - \mu_0)(\mu_0 + \kappa) - \frac{1}{2}\mu(q_\varepsilon^2), \\ q_\varepsilon(t, x)|_{t=0} = \partial_x u_{0\varepsilon}. \end{cases} \quad (5.16)$$

Thanks to (2.4)(up to a slight modification), one has

$$\|2(u_\varepsilon - \mu_0)(\mu_0 + \kappa)\|_{L^\infty} \leq \frac{\sqrt{3}}{3} |\mu_0 + \kappa| \mu_1 \leq (\mu_0 + \kappa)^2 + \frac{1}{12}\mu_1^2.$$

While from (5.6), we deduce that

$$\left\| \frac{1}{2} \mu(q_\varepsilon^2) \right\|_{L^\infty} \leq \frac{1}{2} \mu_1^2.$$

So,

$$\left\| 2(u_\varepsilon - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \mu(q_\varepsilon^2) \right\|_{L^\infty} \leq (\mu_0 + \kappa)^2 + \frac{7}{12} \mu_1^2 = \frac{1}{2} L_0^2. \quad (5.17)$$

Define $Q_\varepsilon(t)$ (for $t > 0$) which solves the following ordinary differential equation

$$\begin{cases} \frac{d}{dt} Q_\varepsilon + \frac{1}{2} (Q_\varepsilon)^2 = \frac{1}{2} L_0^2, \\ Q_\varepsilon(t=0) = \max\{0, \partial_x u_{0\varepsilon}\}. \end{cases} \quad (5.18)$$

Then the function $Q_\varepsilon(t)$ is a supersolution of the parabolic initial-value problem (5.16). The comparison principle for parabolic equations leads to

$$q_\varepsilon(t, x) = \partial_x u_\varepsilon(t, x) \leq Q_\varepsilon(t), \quad \forall t \geq 0, \quad x \in \mathbb{S}. \quad (5.19)$$

While a direct computation yields that $L(t) := \frac{2}{t} + L_0$ (with $t > 0$) satisfies

$$\frac{d}{dt} L(t) + \frac{1}{2} L(t)^2 = \frac{1}{2} L_0^2 + \frac{2L_0}{t} > \frac{1}{2} L_0^2, \quad \forall t > 0,$$

which implies that $L(t)$ is a supersolution of (5.18). Hence, the comparison principle for a parabolic equation yields $Q_\varepsilon(t) \leq L(t)$ for all $t > 0$, which together with (5.19) admits (5.15). \square

Next, we establish the uniform local space-time higher integrability estimate (5.2) motivated by the idea in [36, 41, 44, 45, 46], which is crucial to studying the structures of the Young measure associated with the weak convergence sequence $\partial_x u_\varepsilon$.

Proposition 5.3. *Let $0 < \alpha < 1$, $T > 0$. Then there exists a positive constant C depending only on $\|u_0\|_{H^1}$, T (but independent of ε) such that*

$$\int_0^T \int_{\mathbb{S}} |\partial_x u_\varepsilon(t, x)|^{2+\alpha} dx dt \leq C, \quad (5.20)$$

where $u_\varepsilon = u_\varepsilon(t, x)$ is the unique solution of (5.3).

First, a direct computation yields that

Lemma 5.1. ([7]) *For every $0 < \alpha < 1$, the function $\theta(\xi) = \xi(|\xi| + 1)^\alpha$ with $\xi \in \mathbb{R}$ satisfies the following property.*

$$\theta'(\xi) = ((\alpha + 1)|\xi| + 1) (|\xi| + 1)^{\alpha-1},$$

$$\theta''(\xi) = \alpha(\alpha + 1) \text{sign}(\xi) (|\xi| + 1)^{\alpha-1} + \alpha(1 - \alpha) \text{sign}(\xi) (|\xi| + 1)^{\alpha-2},$$

$$\xi \theta(\xi) - \frac{1}{2} \xi^2 \theta'(\xi) \geq \frac{1 - \alpha}{2} \xi^2 (|\xi| + 1)^\alpha$$

and

$$|\theta(\xi)| \leq |\xi|^{\alpha+1} + |\xi|, \quad |\theta'(\xi)| \leq (\alpha + 1)|\xi| + 1, \quad |\theta''(\xi)| \leq 2\alpha.$$

We are now in a position to prove Proposition 5.3.

Proof of Proposition 5.3. Multiplying the equation in (5.16) by $\theta'(q_\varepsilon)$, we get

$$\begin{aligned} \partial_t \theta(q_\varepsilon) + u_\varepsilon \partial_x \theta(q_\varepsilon) - \varepsilon \theta'(q_\varepsilon) \partial_x^2 q_\varepsilon + \frac{1}{2} \theta'(q_\varepsilon) (q_\varepsilon)^2 \\ = \theta'(q_\varepsilon) \left(2(u_\varepsilon - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \mu(q_\varepsilon^2) \right). \end{aligned}$$

Integrating the above equation over $[0, T] \times \mathbb{S}$, we obtain by integration by parts that

$$\begin{aligned} \int_0^T \int_{\mathbb{S}} [q_\varepsilon \theta(q_\varepsilon) - \frac{1}{2} (q_\varepsilon)^2 \theta'(q_\varepsilon)] dx d\tau \\ = \int_{\mathbb{S}} \theta(q_\varepsilon)(T) dx - \int_{\mathbb{S}} \theta(q_\varepsilon)(0) dx + \varepsilon \int_0^T \int_{\mathbb{S}} (\partial_x q_\varepsilon)^2 \theta''(q_\varepsilon) dx d\tau \\ - \int_0^T \int_{\mathbb{S}} \left(2(u_\varepsilon - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \mu(q_\varepsilon^2) \right) \theta'(q_\varepsilon) dx d\tau. \end{aligned} \quad (5.21)$$

It follows from Lemma 5.1 that

$$\int_0^T \int_{\mathbb{S}} \left(q_\varepsilon \theta(q_\varepsilon) - \frac{1}{2} (q_\varepsilon)^2 \theta'(q_\varepsilon) \right) dx d\tau \geq \frac{1-\alpha}{2} \int_0^T \int_{\mathbb{S}} |q_\varepsilon|^{2+\alpha} dx d\tau \quad (5.22)$$

and

$$\begin{aligned} \left| \int_{\mathbb{S}} \theta(q_\varepsilon)(T) dx \right| &\leq \int_{\mathbb{S}} (|q_\varepsilon(T)|^{1+\alpha} + |q_\varepsilon(T)|) dx \\ &\leq \|q_\varepsilon(T)\|_{L^2(\mathbb{S})}^{1+\alpha} + \|q_\varepsilon(T)\|_{L^2(\mathbb{S})} \leq \mu_1^{1+\alpha} + \mu_1. \end{aligned} \quad (5.23)$$

Similarly, we have

$$\left| \int_{\mathbb{S}} \theta(q_\varepsilon)(0) dx \right| \leq \mu_1^{1+\alpha} + \mu_1. \quad (5.24)$$

On the other hand, thanks to (5.6) and (5.17), together with Lemma 5.1 applied again, we deduce that

$$\varepsilon \int_0^T \int_{\mathbb{S}} (\partial_x q_\varepsilon)^2 \theta''(q_\varepsilon) dx d\tau \leq 2\alpha\varepsilon \|\partial_x q_\varepsilon\|_{L^2([0,T] \times \mathbb{S})}^2 \leq \alpha\mu_1^2 \quad (5.25)$$

and

$$\begin{aligned} \int_0^T \int_{\mathbb{S}} \left(2(u_\varepsilon - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \mu(q_\varepsilon^2) \right) \theta'(q_\varepsilon) dx d\tau \\ \leq \|2(u_\varepsilon - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \mu(q_\varepsilon^2)\|_{L^\infty([0,T] \times \mathbb{S})} \int_0^T \int_{\mathbb{S}} ((1+\alpha)|q_\varepsilon| + 1) dx d\tau \\ \leq \frac{L_0^2}{2} T (1 + (1+\alpha)\mu_1). \end{aligned} \quad (5.26)$$

Therefore, plunging (5.22)-(5.26) into (5.21), we get (5.20), which completes the proof of Proposition 5.3. \square

5.3. Precompactness. In this subsection, we drive the theory of Young measures (see Lemma 4.2 in [41], also [42], [43],[44], [46]) to obtain the necessary compactness of the viscous approximate solution $u_\varepsilon(t, x)$. We first state a compactness lemma.

Lemma 5.2. ([38]) *Let X, B, Y be three Banach spaces and satisfy $X \hookrightarrow B \hookrightarrow Y$, $1 \leq p \leq \infty$, $T > 0$. Assume that a set F of functions f is bounded in $L^p([0, T], X)$ and satisfies that*

$$\|f(\cdot + h) - f(\cdot)\|_{L^p(0, T-h; Y)} \rightarrow 0 \quad \text{as } h \rightarrow 0, \quad \text{uniformly for } f \in F.$$

Then F is relatively compact in $L^p([0, T], B)$ (and in $C([0, T], B)$ if $p = \infty$).

With this compactness lemma in hand, we can prove the weak convergence property in $L^\infty(\mathbb{R}^+; H^1(\mathbb{S}))$.

Proposition 5.4. *There exist a subsequence $\{u_{\varepsilon_j}(t, x), \mu((\partial_x u_{\varepsilon_j}(t, x))^2)\}$ of the sequence $\{u_\varepsilon(t, x), \mu((\partial_x u_\varepsilon(t, x))^2)\}$ and some functions $\{u(t, x), \Pi_1(t)\}$, $u \in L^\infty(\mathbb{R}^+; H^1(\mathbb{S}))$ and $\Pi_1(t) \in L^\infty(\mathbb{R}^+)$, such that*

$$u_{\varepsilon_j} \rightarrow u \quad \text{as } j \rightarrow +\infty \quad \text{uniformly on each compact subset of } \mathbb{R}^+ \times \mathbb{S} \quad (5.27)$$

and

$$\mu((\partial_x u_{\varepsilon_j}(t, x))^2) \rightarrow \Pi_1(t) \quad \text{in } L^p_{loc}(\mathbb{R}^+) \quad \text{as } j \rightarrow +\infty \quad \forall 1 < p < +\infty. \quad (5.28)$$

Proof. According to (5.6), one has that $u_\varepsilon \in L^\infty(\mathbb{R}^+; H^1(\mathbb{S}))$, and u_ε is uniformly bounded in $H^1(\mathbb{S})$. While the first equation in (5.3) yields

$$\partial_t u_\varepsilon = \varepsilon \partial_x^2 u_\varepsilon - u_\varepsilon \partial_x u_\varepsilon - \partial_x P_\varepsilon.$$

Thanks to (1.8), (5.6) and (2.5), together with Hölder's inequality applied, we obtain

$$\|\partial_x P_\varepsilon\|_{L^2(\mathbb{S})} \leq \|\partial_x P_\varepsilon\|_{L^\infty(\mathbb{S})} \leq \frac{9}{4} (\mu_0^2 + (\mu_0 + \kappa)^2) + \left(\frac{1}{4} + \frac{1}{2\pi^2}\right) \mu_1^2,$$

$$\|u_\varepsilon \partial_x u_\varepsilon\|_{L^2(\mathbb{S})} \leq \|u_\varepsilon\|_{L^\infty} \|\partial_x u_\varepsilon\|_{L^2} \leq \mu_1 \sqrt{\mu_0^2 + \frac{1}{4\pi^2} \mu_1^2} \quad \text{and}$$

$$\sqrt{\varepsilon} \|\partial_x^2 u_\varepsilon\|_{L^2([0, T] \times \mathbb{S})} \leq \mu_1 \quad \text{for } \forall T > 0.$$

So, we have

$$\|\partial_t u_\varepsilon\|_{L^2([0, T] \times \mathbb{S})}^2 \leq C(\mu_0, \mu_1, T)$$

with the constant $C(\mu_0, \mu_1, T)$ independent of ε , and consequently,

$$\partial_t u_\varepsilon \in L^2_{loc}(\mathbb{R}^+; L^2(\mathbb{S})).$$

From this, fixing $T > 0$, we get for $0 \leq t, s \leq T$,

$$\|u_\varepsilon(t) - u_\varepsilon(s)\|_{L^2(\mathbb{S})}^2 = \int_{\mathbb{S}} \left(\int_s^t \partial_t u_\varepsilon(\tau, x) d\tau \right)^2 dx \leq |t - s|^{\frac{1}{2}} \|\partial_t u_\varepsilon\|_{L^2([0, T] \times \mathbb{S})}^2.$$

Therefore, applying Lemma 5.2, together with the embedding theorem $H^1(\mathbb{S}) \hookrightarrow C(\mathbb{S}) \hookrightarrow L^2(\mathbb{S})$, we deduce (5.27).

On the other hand, we get from the second equation in (5.6) that

$$\|\mu((\partial_x u_{\varepsilon_j})^2)\|_{L^\infty(\mathbb{R}^+)} \leq \mu_1^2 \quad \text{and}$$

$$\frac{d}{dt} \mu((\partial_x u_{\varepsilon_j})^2) = -2\varepsilon \|\partial_x^2 u_{\varepsilon_j}\|_{L^2}^2, \quad (5.29)$$

which together with (5.6) once again implies

$$\left\| \frac{d}{dt} \mu((\partial_x u_{\varepsilon_j})^2) \right\|_{L^2(\mathbb{R}^+)} \leq \mu_1^2.$$

Therefore, the standard Lions-Aubin's Lemma applied implies (5.28). This completes the proof of Proposition 5.4. \square

Now let $\mu_{t,x}(\lambda)$ be the Young measure associated with $\{q_\varepsilon\}_{\varepsilon>0}$. The theory of the Young measures (see Lemma 4.2 in [41], also [27], [42], [43], [44], [45], [46]) applied implies that, for any continuous function $f = f(\lambda)$ such that $f(\lambda) = o(|\lambda|^r)$ and $\partial_\lambda f(\lambda) = o(|\lambda|^{r-1})$ as $|\lambda| \rightarrow +\infty$ and $r < 2$, and for any $\psi \in L^s(\mathbb{S})$ with $\frac{1}{s} + \frac{r}{2} = 1$, there holds

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{S}} f(q_\varepsilon(t, x)) \psi(x) dx = \int_{\mathbb{S}} \overline{f(q)} \psi(x) dx \quad (5.30)$$

uniformly in every compact subset of \mathbb{R}^+ . Here

$$\overline{f(q)} := \int_{\mathbb{S}} f(x) d\mu_{t,x}(\lambda) \in C([0, \infty); L^{r'/r}(\mathbb{S}))$$

with $r' \in (r, 2)$. Moreover, for all $T > 0$, there hold

$$\lim_{\varepsilon \rightarrow 0} \int_0^T \int_{\mathbb{S}} g(q_\varepsilon) \varphi dx dt = \int_0^T \int_{\mathbb{S}} \overline{g(q)} \varphi dx dt$$

and

$$\lambda \in L_{loc}^\ell(\mathbb{R}^+ \times \mathbb{S} \times \mathbb{R}, dt \otimes dx \otimes d\mu_{t,x}(\lambda)) \quad \text{for all } \ell < 3,$$

where $g = g(t, x, \lambda)$ is a continuous function satisfying $g = o(|\lambda|^\ell)$ as $|\lambda| \rightarrow +\infty$ for some $\ell < 3$, and with $\frac{\ell}{3} + \frac{1}{m} < 1$. And also

$$\lambda \in L^\infty(\mathbb{R}^+; L^2(\mathbb{S} \times \mathbb{R}, dx \otimes d\mu_{t,x}(\lambda))) \quad \text{and} \quad \overline{q}(t, x) = \partial_x u(t, x). \quad (5.31)$$

We are in a position to study the structure of the Young measure $\mu_{t,x}(\lambda)$.

Lemma 5.3. *Let $E = E(\lambda) \in W^{2,\infty}(\mathbb{R})$ be a given convex function satisfying $E(\lambda) = O(|\lambda|)$ and the first derivative $DE(\lambda) = O(1)$ for $|\lambda| \rightarrow +\infty$. Then there holds*

$$\partial_t \overline{E(q)} + \partial_x (u \overline{E(q)}) \leq \overline{qE(q)} - \frac{1}{2} \overline{q^2 DE(q)} + \overline{DE(q)} [2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1] \quad (5.32)$$

in the sense of distributions on $\mathbb{R}^+ \times \mathbb{S}$.

Proof. Multiplying the first equation in (5.16) by $DE(q_\varepsilon)$, we get

$$\begin{aligned} \partial_t E(q_\varepsilon) + u_\varepsilon \partial_x E(q_\varepsilon) - \varepsilon DE(q_\varepsilon) \partial_x^2 q_\varepsilon + \frac{1}{2} DE(q_\varepsilon) (q_\varepsilon)^2 \\ = DE(q_\varepsilon) \left(2(u_\varepsilon - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \mu(q_\varepsilon^2) \right), \end{aligned} \quad (5.33)$$

which implies

$$\begin{aligned} \partial_t E(q_\varepsilon) + \partial_x (u_\varepsilon E(q_\varepsilon)) &= q_\varepsilon E(q_\varepsilon) + \varepsilon \partial_x (DE(q_\varepsilon) \partial_x q_\varepsilon) - \varepsilon D^2 E(q_\varepsilon) (\partial_x q_\varepsilon)^2 \\ &\quad - \frac{1}{2} DE(q_\varepsilon) (q_\varepsilon)^2 + DE(q_\varepsilon) \left(2(u_\varepsilon - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \mu(q_\varepsilon^2) \right). \end{aligned}$$

Noting that $\sqrt{\varepsilon} \partial_x q_\varepsilon$ is uniformly bounded in $L^2(\mathbb{R}^+ \times \mathbb{S})$ (according to (5.6)), and taking the limit $\varepsilon \rightarrow 0$, one obtains from Proposition 5.4 and (5.31) that (5.32) holds. \square

Taking $E(\lambda) = \lambda$ in (5.33) gives

$$\partial_t q_\varepsilon + \partial_x (u_\varepsilon q_\varepsilon) = \varepsilon \partial_x^2 q_\varepsilon + \frac{1}{2} (q_\varepsilon)^2 + \left(2(u_\varepsilon - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \mu(q_\varepsilon^2) \right).$$

Similar to the proof of Lemma 5.3, we may get

Lemma 5.4. *There holds*

$$\partial_t \bar{q} + u \partial_x \bar{q} = \left(\frac{1}{2} \bar{q}^2 - \bar{q}^2 \right) + \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) \quad (5.34)$$

in the sense of distributions on $\mathbb{R}^+ \times \mathbb{S}$.

Lemma 5.5. *Let $E = E(\lambda) \in W^{2,\infty}(\mathbb{R})$ be a given convex function satisfying $E(\lambda) = O(|\lambda|)$ and $DE(\lambda) = O(1)$ for $|\lambda| \rightarrow +\infty$. Then there holds*

$$\begin{aligned} & \partial_t (\overline{E(q)} - E(\bar{q})) + \partial_x (u(\overline{E(q)} - E(\bar{q}))) \\ & \leq \int_{\mathbb{R}} \left(\lambda E(\lambda) - \frac{1}{2} DE(\lambda) \lambda^2 \right) d\mu_{t,x}(\lambda) + \frac{1}{2} DE(\bar{q}) (\bar{q})^2 - \bar{q} E(\bar{q}) \\ & \quad - \frac{1}{2} DE(\bar{q}) (\bar{q}^2 - (\bar{q})^2) + (\overline{DE(q)} - DE(\bar{q})) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) \end{aligned} \quad (5.35)$$

in the sense of distributions on $\mathbb{R}^+ \times \mathbb{S}$.

Proof. We first get from (5.34) that

$$\partial_t \bar{q} + u \partial_x \bar{q} = \left(\frac{1}{2} \bar{q}^2 - \bar{q}^2 \right) + \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) \quad (5.36)$$

Taking the convolution of (5.36) with the standard Friedrichs mollifier, $j_\delta(x)$, one gets that

$$\partial_t \bar{q}^\delta + u \partial_x \bar{q}^\delta = j_\delta * \left(\left(\frac{1}{2} \bar{q}^2 - \bar{q}^2 \right) + \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) \right) + r_\delta, \quad (5.37)$$

where $\bar{q}^\delta = j_\delta * \bar{q}$, $r_\delta = u \partial_x \bar{q}^\delta - j_\delta * (u \partial_x \bar{q})$. Multiplying (5.37) by $DE(\bar{q}^\delta)$ gives rise to

$$\begin{aligned} & \partial_t E(\bar{q}^\delta) + \partial_x (u E(\bar{q}^\delta)) = \bar{q} E(\bar{q}^\delta) \\ & \quad + DE(\bar{q}^\delta) \left\{ j_\delta * \left(\left(\frac{1}{2} \bar{q}^2 - \bar{q}^2 \right) + \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) \right) + r_\delta \right\}. \end{aligned} \quad (5.38)$$

Taking the limit $\delta \rightarrow 0^+$ in (5.38) and using the fact that,

$$r_\delta \rightarrow 0 \quad \text{as } \delta \rightarrow 0^+ \quad \text{in } L^1_{loc}(\mathbb{R}^+, L^1(\mathbb{S})),$$

which follows from lemma II.1 of [21], one obtains that

$$\begin{aligned} & \partial_t E(\bar{q}) + \partial_x (u E(\bar{q})) \\ & = \bar{q} E(\bar{q}) + DE(\bar{q}) \left(\frac{1}{2} \bar{q}^2 - (\bar{q})^2 - \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) \right). \end{aligned} \quad (5.39)$$

Subtracting (5.39) from (5.32) yields (5.35). \square

Lemma 5.6. *For each $R > 0$,*

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{S}} (\overline{Q_R^\pm(q)}(t, x) - Q_R^\pm(\bar{q})(t, x)) dx = 0, \quad (5.40)$$

where

$$Q_R(\lambda) = \begin{cases} \frac{1}{2} \lambda^2 & \text{if } |\lambda| \leq R, \\ R|\lambda| - \frac{1}{2} R^2 & \text{if } |\lambda| \geq R, \end{cases}$$

and $Q_R^+(\lambda) := \mathbf{1}_{\lambda \geq 0} Q_R(\lambda)$, $Q_R^-(\lambda) := \mathbf{1}_{\lambda \leq 0} Q_R(\lambda)$ for $\lambda \in \mathbb{R}$, where $\mathbf{1}_A$ denotes the characteristic function of the set A .

Proof. Thanks to the definition of $Q_R(\lambda)$ and $Q_R^\pm(\lambda)$, one may verify that $Q_R^\pm(\lambda)$ and $Q_R(\lambda) = Q_R^+(\lambda) + Q_R^-(\lambda)$ all satisfy the assumptions on E in Lemma 5.5, so one can apply (5.35) to all of them.

Note that $Q_R(\lambda)$ is a convex function, we get from Jensen's inequality that

$$\begin{aligned} 0 &\leq \overline{Q_R(q)}(t, x) - Q_R(\bar{q}) \\ &= \frac{1}{2} \left(\overline{q^2} - (\bar{q})^2 \right) - \frac{1}{2} \left(\int_{\mathbb{R}} (|\lambda| - R)^2 \mathbf{1}_{|\lambda| \geq R} d\mu_{t,x}(\lambda) - (|\bar{q}| - R)^2 \mathbf{1}_{|\bar{q}| \geq R} \right). \end{aligned}$$

While $(|\lambda| - R)^2 \mathbf{1}_{|\lambda| \geq R}$ is a convex function, one gets that

$$\int_{\mathbb{R}} (|\lambda| - R)^2 \mathbf{1}_{|\lambda| \geq R} d\mu_{t,x}(\lambda) - (|\bar{q}| - R)^2 \mathbf{1}_{|\bar{q}| \geq R} \geq 0.$$

Hence,

$$0 \leq \overline{Q_R^\pm(q)} - Q_R^\pm(\bar{q}) \leq \overline{Q_R(q)} - Q_R(\bar{q}) \leq \frac{1}{2} (\overline{q^2} - (\bar{q})^2). \quad (5.41)$$

On the other hand, thanks to the fact that $u \in C(\mathbb{R}^+ \times \mathbb{S})$ and (5.31), we get for each test function $\phi \in C^\infty(\mathbb{S})$

$$\begin{aligned} \lim_{t \rightarrow 0^+} \int_{\mathbb{S}} q(t, x) \phi(x) dx &= - \lim_{t \rightarrow 0^+} \int_{\mathbb{S}} u(t, x) \partial_x \phi(x) dx \\ &= - \int_{\mathbb{S}} u_0(x) \partial_x \phi(x) dx = \int_{\mathbb{S}} q_0(x) \phi(x) dx. \end{aligned}$$

From this, together with the fact that $q_\varepsilon \in C(\mathbb{R}^+; L^2(\mathbb{S})) \cap L^\infty(\mathbb{R}^+; L^2(\mathbb{S}))$ is uniformly bounded with respect to $\varepsilon > 0$, we obtain that

$$\bar{q}(t, x) \rightharpoonup q_0(x) = \partial_x u_0 \quad \text{as } t \rightarrow 0^+ \quad \text{in } L^2(\mathbb{S}),$$

and so

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{S}} (\bar{q}(t, x))^2 dx \geq \int_{\mathbb{S}} (q_0(x))^2 dx.$$

While the energy estimate (5.6) together with (5.30) implies that

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{S}} (\bar{q}(t, x))^2 dx \leq \lim_{t \rightarrow 0^+} \int_{\mathbb{S}} \overline{q(t, x)^2} dx \leq \int_{\mathbb{S}} (q_0(x))^2 dx.$$

Hence, we have

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{S}} (\bar{q}(t, x))^2 dx = \lim_{t \rightarrow 0^+} \int_{\mathbb{S}} \overline{q(t, x)^2} dx = \int_{\mathbb{S}} (q_0(x))^2 dx,$$

which along with (5.41) implies (5.40). \square

We are in a position to prove that the Young measure $\mu_{t,x}(\lambda)$ is a Dirac measure.

Lemma 5.7. *Let $\mu_{t,x}(\lambda)$ be the Young measure associated with $\{q_\varepsilon\}_{\varepsilon > 0}$. Then*

$$\mu_{t,x}(\lambda) = \delta_{\bar{q}(t,x)}(\lambda) \quad \forall \quad a.e. \quad (t, x) \in \mathbb{R}^+ \times \mathbb{S}. \quad (5.42)$$

Proof. We first apply (5.35) to $E(\lambda) = Q_R^+(\lambda)$ to obtain

$$\begin{aligned} & \partial_t(\overline{Q_R^+(q)} - Q_R^+(\bar{q})) + \partial_x \left(u \left(\overline{Q_R^+(q)} - Q_R^+(\bar{q}) \right) \right) \\ & \leq \frac{R}{2} \left(\int_{\mathbb{R}} \lambda(\lambda - R) \mathbf{1}_{\lambda \geq R} d\mu_{t,x}(\lambda) - \bar{q}(\bar{q} - R) \mathbf{1}_{\bar{q} \geq R} \right) \\ & \quad - \frac{1}{2} DQ_R^+(\bar{q})(\bar{q}^2 - (\bar{q})^2) + (\overline{DQ_R^+(q)} - DQ_R^+(\bar{q})) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right). \end{aligned} \quad (5.43)$$

Note that both $\bar{q}(t, x)$ and $q_\varepsilon(t, x)$ are bounded above by $\frac{2}{t} + C$ with $C = L_0$ in (5.15). Thus, $\text{Supp } \mu_{t,x}(\cdot) \subset (-\infty, \frac{2}{t} + C)$. Therefore, for $R \geq \frac{2}{t} + C$, i. e., $t \geq \frac{2}{R-C}$ (for $R > C$), one gets from (5.43) that

$$\begin{aligned} & \partial_t(\overline{Q_R^+(q)} - Q_R^+(\bar{q})) + \partial_x \left(u \left(\overline{Q_R^+(q)} - Q_R^+(\bar{q}) \right) \right) \\ & \leq (\overline{DQ_R^+(q)} - DQ_R^+(\bar{q})) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right), \end{aligned}$$

which implies for $t \geq \frac{2}{R-C}$ that

$$\begin{aligned} & \int_{\mathbb{S}} (\overline{Q_R^+(q)} - Q_R^+(\bar{q}))(t, x) dx \leq \int_{\mathbb{S}} (\overline{Q_R^+(q)} - Q_R^+(\bar{q})) \left(\frac{2}{R-C}, x \right) dx \\ & \quad + \int_{\frac{2}{R-C}}^t \int_{\mathbb{S}} (\overline{DQ_R^+(q)} - DQ_R^+(\bar{q})) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) dx ds. \end{aligned} \quad (5.44)$$

For any f , define $f_+ := \max\{f, 0\}$, $f_- := \min\{f, 0\}$. Using this notation, together with the definition of $Q_R^+(q)$, one has

$$\begin{aligned} \overline{Q_R^+(q)} - Q_R^+(\bar{q}) &= \frac{1}{2} (\overline{q_+^2} - (\bar{q}_+)^2) - \frac{1}{2} \left\{ \int_{\mathbb{R}} (\lambda - R)^2 \mathbf{1}_{\lambda \geq R} d\mu_{t,x}(\lambda) - (\bar{q} - R)^2 \mathbf{1}_{\bar{q} \geq R} \right\} \\ &= \frac{1}{2} (\overline{q_+^2} - (\bar{q}_+)^2) \end{aligned}$$

and

$$\overline{DQ_R^+(q)} - DQ_R^+(\bar{q}) = (\overline{q_+} - \bar{q}_+) - \left\{ \int_{\mathbb{R}} (\lambda - R) \mathbf{1}_{\lambda \geq R} d\mu_{t,x}(\lambda) - (\bar{q} - R) \mathbf{1}_{\bar{q} \geq R} \right\},$$

which applied to (5.44) gives rise to

$$\begin{aligned} & \int_{\mathbb{S}} (\overline{q_+^2} - (\bar{q}_+)^2)(t, x) dx \leq 2 \int_{\frac{2}{R-C}}^t \int_{\mathbb{S}} (\overline{q_+} - \bar{q}_+) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) dx ds \\ & \quad + \int_{\mathbb{S}} (\overline{q_+^2} - (\bar{q}_+)^2) \left(\frac{2}{R-C}, x \right) dx - 2 \int_{\frac{2}{R-C}}^t \int_{\mathbb{S}} \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) \\ & \quad \times \left(\int_{\mathbb{R}} (\lambda - R) \mathbf{1}_{\lambda \geq R} d\mu_{t,x}(\lambda) - (\bar{q} - R) \mathbf{1}_{\bar{q} \geq R} \right) dx ds. \end{aligned}$$

Taking the limit $R \rightarrow +\infty$ and using (5.31), (5.40), and the Lebesgue dominated convergence theorem, we conclude that for all $t > 0$

$$\int_{\mathbb{S}} (\overline{q_+^2} - (\bar{q}_+)^2)(t, x) dx \leq 2 \int_0^t \int_{\mathbb{S}} (\overline{q_+} - \bar{q}_+) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) dx ds. \quad (5.45)$$

Since $\overline{q^2} - (\overline{q})^2 = (\overline{q_+^2} - (\overline{q_+})^2) + (\overline{q_-^2} - (\overline{q_-})^2)$, it remains to estimate the part associated with $(\overline{q_-^2} - (\overline{q_-})^2)$, which may be approximated by $\overline{Q_R^-(q)} - Q_R^-(\overline{q})$ as R goes to $+\infty$.

Indeed, we apply (5.35) to $E(\lambda) = Q_R^-(\lambda)$ to obtain

$$\begin{aligned} & \partial_t (\overline{Q_R^-(q)} - Q_R^-(\overline{q})) + \partial_x (u(\overline{Q_R^-(q)} - Q_R^-(\overline{q}))) \\ & \leq -\frac{R}{2} \left\{ \int_{\mathbb{R}} \lambda(\lambda + R) \mathbf{1}_{\lambda \leq -R} d\mu_{t,x}(\lambda) - \overline{q}(\overline{q} + R) \mathbf{1}_{\overline{q} \leq -R} \right\} \\ & \quad - \frac{1}{2} DQ_R^-(\overline{q}) (\overline{q^2} - (\overline{q})^2) + (\overline{DQ_R^-(q)} - DQ_R^-(\overline{q})) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right). \end{aligned}$$

Hence, integrating this inequality over $[0, t) \times \mathbb{S}$ and using (5.40), we get that

$$\begin{aligned} & \int_{\mathbb{S}} (\overline{Q_R^-(q)} - Q_R^-(\overline{q}))(t, x) dx \leq \frac{R}{2} \int_0^t \int_{\mathbb{S}} (\overline{q^2} - (\overline{q})^2) dx ds \\ & \quad - \frac{R}{2} \int_0^t \int_{\mathbb{S}} \left\{ \int_{\mathbb{R}} \lambda(\lambda + R) \mathbf{1}_{\lambda \leq -R} d\mu_{t,x}(\lambda) - \overline{q}(\overline{q} + R) \mathbf{1}_{\overline{q} \leq -R} \right\} dx ds \quad (5.46) \\ & \quad + \int_0^t \int_{\mathbb{S}} (\overline{DQ_R^-(q)} - DQ_R^-(\overline{q})) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) dx ds. \end{aligned}$$

While a direct computation yields

$$\begin{aligned} & \overline{Q_R^-(q)} - Q_R^-(\overline{q}) \\ & = \frac{1}{2} (\overline{q_-^2} - (\overline{q_-})^2) - \frac{1}{2} \left\{ \int_{\mathbb{R}} (\lambda + R)^2 \mathbf{1}_{\lambda \leq -R} d\mu_{t,x}(\lambda) - (\overline{q} + R)^2 \mathbf{1}_{\overline{q} \leq -R} \right\}, \end{aligned}$$

which together with (5.45) and (5.46) leads to

$$\begin{aligned} & \int_{\mathbb{S}} \left(\frac{1}{2} (\overline{q_+^2} - (\overline{q_+})^2) + \overline{Q_R^-(q)} - Q_R^-(\overline{q}) \right) (t, x) dx \\ & \leq R \int_0^t \int_{\mathbb{S}} \left(\frac{1}{2} (\overline{q_+^2} - (\overline{q_+})^2) + \overline{Q_R^-(q)} - Q_R^-(\overline{q}) \right) (s, x) dx ds \\ & \quad + \frac{R}{2} \int_0^t \int_{\mathbb{S}} \left\{ \int_{\mathbb{R}} R(\lambda + R) \mathbf{1}_{\lambda \leq -R} d\mu_{t,x}(\lambda) - R(\overline{q} + R) \mathbf{1}_{\overline{q} \leq -R} \right\} dx ds \\ & \quad + \int_0^t \int_{\mathbb{S}} (\overline{DQ_R^-(q)} - DQ_R^-(\overline{q}) + \overline{q_+} - \overline{q_+}) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) dx ds. \quad (5.47) \end{aligned}$$

Note that

$$\begin{aligned} 0 & \leq \overline{DQ_R^-(q)} - DQ_R^-(\overline{q}) + \overline{q_+} - \overline{q_+} \\ & = - \left(\int_{\mathbb{R}} (\lambda + R) \mathbf{1}_{\lambda \leq -R} d\mu_{t,x}(\lambda) - (\overline{q} + R) \mathbf{1}_{\overline{q} \leq -R} \right). \end{aligned}$$

Let $L > 0$ (for example, taking $L = L_0^2$ in (5.15)) be a constant such that $\|2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \mu(q_\varepsilon^2)\|_{L^\infty} \leq \frac{L}{2}$ (see (5.11)). Then,

$$\begin{aligned} & \int_0^t \int_{\mathbb{S}} (\overline{DQ_R^-(q)} - DQ_R^-(\overline{q}) + \overline{q_+} - \overline{q_+}) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2} \Pi_1 \right) dx ds \\ & \leq \frac{R}{2} \int_0^t \int_{\mathbb{S}} \left(\int_{\mathbb{R}} \frac{L}{R} (\lambda + R) \mathbf{1}_{\lambda \leq -R} d\mu_{t,x}(\lambda) - \frac{L}{R} (\overline{q} + R) \mathbf{1}_{\overline{q} \leq -R} \right) dx ds. \end{aligned}$$

Therefore, for $R \geq \sqrt{L}$, we get from the fact that $(\lambda + R)\mathbf{1}_{\lambda \leq -R}$ is a concave function that

$$\begin{aligned} & \frac{R}{2} \int_0^t \int_{\mathbb{S}} \left(\int_{\mathbb{R}} R(\lambda + R)\mathbf{1}_{\lambda \leq -R} d\mu_{t,x}(\lambda) - R(\bar{q} + R)\mathbf{1}_{\bar{q} \leq -R} \right) dx ds \\ & + \int_0^t \int_{\mathbb{S}} \left(\overline{DQ_R^-(q)} - DQ_R^-(\bar{q}) + \bar{q}_+ - q_+ \right) \left(2(u - \mu_0)(\mu_0 + \kappa) - \frac{1}{2}\Pi_1 \right) dx ds \\ & \leq \frac{R}{2} \int_0^t \int_{\mathbb{S}} \left(\int_{\mathbb{R}} (R - \frac{L}{R})(\lambda + R)\mathbf{1}_{\lambda \leq -R} d\mu_{t,x}(\lambda) - (R - \frac{L}{R})(\bar{q} + R)\mathbf{1}_{\bar{q} \leq -R} \right) dx ds \\ & \leq 0. \end{aligned} \tag{5.48}$$

It follows from (5.47), (5.48) and Gronwall's inequality that

$$\int_{\mathbb{S}} \left(\frac{1}{2}(\bar{q}_+^2 - (\bar{q}_+)^2) + \overline{Q_R^-(q)} - Q_R^-(\bar{q}) \right) (t, x) dx = 0, \quad \forall \quad t \geq 0. \tag{5.49}$$

Thus, by Fatou's lemma, one can take the limit as $R \rightarrow +\infty$ in (5.49) to conclude that

$$\int_{\mathbb{S}} (\bar{q}^2 - (\bar{q})^2)(t, x) dx \leq 0, \quad \forall \quad t \geq 0.$$

From this, together with the fact $(\bar{q})^2 \leq \bar{q}^2$, we get

$$\int_{\mathbb{S}} \bar{q}^2(t, x) dx = \int_{\mathbb{S}} (\bar{q})^2(t, x) dx, \quad \forall \quad t \geq 0,$$

which implies (5.42). \square

5.4. Proof of Theorem 5.1.

Proof of Theorem 5.1. With all the preparations given in the previous subsection, we are in a position to conclude the proof of the theorem. Let $u(t, x)$ be the limit of the viscous approximate solutions $u_\varepsilon(t, x)$ as $\varepsilon \rightarrow 0^+$. It then follows from Propositions 5.1, 5.2 and 5.4 that $u(t, x) \in C(\mathbb{R}^+ \times \mathbb{S}) \cap L^\infty(\mathbb{R}^+, H^1(\mathbb{S}))$, $\Pi(t) \in L^\infty(\mathbb{R}^+)$ and (1.9) (5.1) hold.

Now we claim that

$$q_\varepsilon = \partial_x u_\varepsilon \rightarrow q = \partial_x u \quad \text{as } \varepsilon \rightarrow 0^+ \quad \text{in } L_{loc}^2(\mathbb{R}^+ \times \mathbb{S}). \tag{5.50}$$

Indeed, it follows from (5.31) and Lemma 5.7 that there exists a subsequence of $\{u_\varepsilon(t, x)\}$, still denoted by itself, such that

$$q_\varepsilon = \partial_x u_\varepsilon \rightarrow q = \partial_x u \quad \text{in } L_{loc}^{p_1}(\mathbb{R}^+, L^{p_2}(\mathbb{S})) \quad \forall \quad p_1 < \infty, \quad p_2 < 2.$$

This together with Proposition 5.3 and a standard interpolation theorem applied implies

$$q_\varepsilon = \partial_x u_\varepsilon \rightarrow q = \partial_x u \quad \text{in } L_{loc}^p(\mathbb{R}^+ \times \mathbb{S}) \quad \forall \quad p < 3, \tag{5.51}$$

which gives (5.50).

Thus, we get from (5.29) and (5.6) that $\Pi(t) = \mu((\partial_x u)^2)$.

Taking $\varepsilon \rightarrow 0^+$ in (5.3), one finds from (5.50) and Proposition 5.4 that u is an admissible weak solution to (1.2). It then follows from (5.51) that $\partial_x u \in L_{loc}^p(\mathbb{R}^+ \times \mathbb{S})$ for any $1 \leq p < 3$. Hence the local space-time higher integrability estimate (5.2) holds. This completes the proof of Theorem 5.1. \square

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